1. For the value of x under consideration we have

   (a) $x = (3 + x - 2x^2)^{1/4} \Leftrightarrow x^4 = 3 + x - 2x^2 \Leftrightarrow f(x) = 0$

   (b) $x = \left(\dfrac{x + 3 - x^4}{2}\right)^{1/2} \Leftrightarrow 2x^2 = x + 3 - x^4 \Leftrightarrow f(x) = 0$

   (c) $x = \left(\dfrac{x + 3}{x^2 + 2}\right)^{1/2} \Leftrightarrow x^2(x^2 + 2) = x + 3 \Leftrightarrow f(x) = 0$

   (d) $x = \dfrac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1} \Leftrightarrow 4x^4 + 4x^2 - x = 3x^4 + 2x^2 + 3 \Leftrightarrow f(x) = 0$

2. (a) $p_4 = 1.10782$; (b) $p_4 = 0.987506$; (c) $p_4 = 1.12364$; (d) $p_4 = 1.12412$;

   (b) Part (d) gives the best answer since $|p_4 - p_3|$ is the smallest for (d).

3. The sequence in (c) converges faster than in (d). The sequences in (a) and (b) diverge.

4. The order in descending speed of convergence is (b), (d), and (a). The sequence in (c) does not converge.

5. With $g(x) = \sqrt{1 + \frac{1}{x}}$ and $p_0 = 1$, we have $p_4 = 1.324$.

6. With $g(x) = (3x^2 + 3)^{1/4}$ and $p_0 = 1$, $p_6 = 1.94332$ is accurate to within 0.01.

7. Since $g'(x) = \frac{1}{4}\cos\frac{x}{2}$, g is continuous and g' exists on $[0, 2\pi]$. Further, $g'(x) = 0$ only when $x = \pi$, so that $g(0) = g(2\pi) = \pi \le g(x) =\le g(\pi) = \pi + \frac{1}{2}$ and $|g'(x)| \le \frac{1}{4}$, for $0 \le x \le 2\pi$. Theorem 2.3 implies that a unique fixed point p exists in $[0, 2\pi]$. With $k = \frac{1}{4}$ and $p_0 = \pi$, we have $p_1 = \pi + \frac{1}{2}$. Corollary 2.5 implies that

$$|p_n - p| \le \frac{k^n}{1-k}|p_1 - p_0| = \frac{2}{3}\left(\frac{1}{4}\right)^n.$$

   For the bound to be less than 0.1, we need $n \ge 4$. However, $p_3 = 3.626996$ is accurate to within 0.01.

8. Using $p_0 = 1$ gives $p_{12} = 0.6412053$. Since $|g'(x)| = 2^{-x}\ln 2 \le 0.551$ on $\left[\frac{1}{3}, 1\right]$ with $k = 0.551$, Corollary 2.5 gives a bound of 16 iterations.

9. For $p_0 = 1.0$ and $g(x) = 0.5(x + \frac{3}{x})$, we have $\sqrt{3} \approx p_4 = 1.73205$.

10. For $g(x) = 5/\sqrt{x}$ and $p_0 = 2.5$, we have $p_{14} = 2.92399$.

11. (a) With $[0, 1]$ and $p_0 = 0$, we have $p_9 = 0.257531$.
    (b) With $[2.5, 3.0]$ and $p_0 = 2.5$, we have $p_{17} = 2.690650$.
    (c) With $[0.25, 1]$ and $p_0 = 0.25$, we have $p_{14} = 0.909999$.
    (d) With $[0.3, 0.7]$ and $p_0 = 0.3$, we have $p_{39} = 0.469625$.
    (e) With $[0.3, 0.6]$ and $p_0 = 0.3$, we have $p_{48} = 0.448059$.
    (f) With $[0, 1]$ and $p_0 = 0$, we have $p_6 = 0.704812$.

12. The inequalities in Corollary 2.4 give $|p_n - p| < k^n \max(p_0 - a, b - p_0)$. We want

$$k^n \max(p_0 - a, b - p_0) < 10^{-5} \quad \text{so we need} \quad n > \frac{\ln(10^{-5}) - \ln(\max(p_0 - a, b - p_0))}{\ln k}.$$

    (a) Using $g(x) = 2 + \sin x$ we have $k = 0.9899924966$ so that with $p_0 = 2$ we have $n > \ln(0.00001)/\ln k = 1144.663221$. However, our tolerance is met with $p_{63} = 2.5541998$.
    (b) Using $g(x) = \sqrt[3]{2x+5}$ we have $k = 0.1540802832$ so that with $p_0 = 2$ we have $n > \ln(0.00001)/\ln k = 6.155718005$. However, our tolerance is met with $p_6 = 2.0945503$.
    (c) Using $g(x) = \sqrt{e^x/3}$ and the interval $[0, 1]$ we have $k = 0.4759448347$ so that with $p_0 = 1$ we have $n > \ln(0.00001)/\ln k = 15.50659829$. However, our tolerance is met with $p_{12} = 0.91001496$.
    (d) Using $g(x) = \cos x$ and the interval $[0, 1]$ we have $k = 0.8414709848$ so that with $p_0 = 0$ we have $n > \ln(0.00001)/\ln k > 66.70148074$. However, our tolerance is met with $p_{30} = 0.73908230$.

13. For $g(x) = (2x^2 - 10\cos x)/(3x)$, we have the following:

$$p_0 = 3 \Rightarrow p_8 = 3.16193; \quad p_0 = -3 \Rightarrow p_8 = -3.16193.$$

For $g(x) = \arccos(-0.1x^2)$, we have the following:

$$p_0 = 1 \Rightarrow p_{11} = 1.96882; \quad p_0 = -1 \Rightarrow p_{11} = -1.96882.$$

14. For $g(x) = \dfrac{1}{\tan x} - \dfrac{1}{x} + x$ and $p_0 = 4$, we have $p_4 = 4.493409$.

15. With $g(x) = \dfrac{1}{\pi} \arcsin\left(-\dfrac{x}{2}\right) + 2$, we have $p_5 = 1.683855$.

16. (a) If fixed-point iteration converges to the limit p, then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} 2p_{n-1} - Ap_{n-1}^2 = 2p - Ap^2.$$

Solving for p gives $p = \dfrac{1}{A}$.

(b) Any subinterval $[c,d]$ of $\left(\dfrac{1}{2A}, \dfrac{3}{2A}\right)$ containing $\dfrac{1}{A}$ suffices.

Since

$$g(x) = 2x - Ax^2, \quad g'(x) = 2 - 2Ax,$$

so $g(x)$ is continuous, and $g'(x)$ exists. Further, $g'(x) = 0$ only if $x = \dfrac{1}{A}$.

Since

$$g\left(\frac{1}{A}\right) = \frac{1}{A}, \quad g\left(\frac{1}{2A}\right) = g\left(\frac{3}{2A}\right) = \frac{3}{4A}, \quad \text{and we have} \quad \frac{3}{4A} \le g(x) \le \frac{1}{A}.$$

For x in $\left(\frac{1}{2A}, \frac{3}{2A}\right)$, we have

$$\left|x - \frac{1}{A}\right| < \frac{1}{2A} \quad \text{so} \quad |g'(x)| = 2A\left|x - \frac{1}{A}\right| < 2A\left(\frac{1}{2A}\right) = 1.$$

17. One of many examples is $g(x) = \sqrt{2x-1}$ on $\left[\frac{1}{2}, 1\right]$.

18. (a) The proof of existence is unchanged. For uniqueness, suppose p and q are fixed points in $[a,b]$ with $p \neq q$. By the Mean Value Theorem, a number ξ in (a,b) exists with

$$p - q = g(p) - g(q) = g'(\xi)(p-q) \le k(p-q) < p - q,$$

giving the same contradiction as in Theorem 2.3.

(b) Consider $g(x) = 1 - x^2$ on $[0,1]$. The function g has the unique fixed point

$$p = \frac{1}{2}\left(-1 + \sqrt{5}\right).$$

With $p_0 = 0.7$, the sequence eventually alternates between 0 and 1.

19. (a) Suppose that $x_0 > \sqrt{2}$. Then

$$x_1 - \sqrt{2} = g(x_0) - g\left(\sqrt{2}\right) = g'(\xi)\left(x_0 - \sqrt{2}\right),$$

where $\sqrt{2} < \xi < x_0$. Thus, $x_1 - \sqrt{2} > 0$ and $x_1 > \sqrt{2}$. Further,

$$x_1 = \frac{x_0}{2} + \frac{1}{x_0} < \frac{x_0}{2} + \frac{1}{\sqrt{2}} = \frac{x_0 + \sqrt{2}}{2}$$

and $\sqrt{2} < x_1 < x_0$. By an inductive argument,

$$\sqrt{2} < x_{m+1} < x_m < \ldots < x_0.$$

Thus, $\{x_m\}$ is a decreasing sequence which has a lower bound and must converge. Suppose $p = \lim_{m\to\infty} x_m$. Then

$$p = \lim_{m\to\infty}\left(\frac{x_{m-1}}{2} + \frac{1}{x_{m-1}}\right) = \frac{p}{2} + \frac{1}{p}. \quad \text{Thus} \quad p = \frac{p}{2} + \frac{1}{p},$$

which implies that $p = \pm\sqrt{2}$. Since $x_m > \sqrt{2}$ for all m, we have $\lim_{m\to\infty} x_m = \sqrt{2}$.

(b) We have

$$0 < \left(x_0 - \sqrt{2}\right)^2 = x_0^2 - 2x_0\sqrt{2} + 2,$$

so $2x_0\sqrt{2} < x_0^2 + 2$ and $\sqrt{2} < \frac{x_0}{2} + \frac{1}{x_0} = x_1$.

(c) Case 1: $0 < x_0 < \sqrt{2}$, which implies that $\sqrt{2} < x_1$ by part (b). Thus,

$$0 < x_0 < \sqrt{2} < x_{m+1} < x_m < \ldots < x_1 \quad \text{and} \quad \lim_{m\to\infty} x_m = \sqrt{2}.$$

Case 2: $x_0 = \sqrt{2}$, which implies that $x_m = \sqrt{2}$ for all m and $\lim_{m\to\infty} x_m = \sqrt{2}$.
Case 3: $x_0 > \sqrt{2}$, which by part (a) implies that $\lim_{m\to\infty} x_m = \sqrt{2}$.

20. (a) Let

$$g(x) = \frac{x}{2} + \frac{A}{2x}.$$

Note that $g\left(\sqrt{A}\right) = \sqrt{A}$. Also,

$$g'(x) = 1/2 - A/\left(2x^2\right) \text{ if } x \neq 0 \quad \text{and} \quad g'(x) > 0 \text{ if } x > \sqrt{A}.$$

If $x_0 = \sqrt{A}$, then $x_m = \sqrt{A}$ for all m and $\lim_{m\to\infty} x_m = \sqrt{A}$.
If $x_0 > A$, then

$$x_1 - \sqrt{A} = g(x_0) - g\left(\sqrt{A}\right) = g'(\xi)\left(x_0 - \sqrt{A}\right) > 0.$$

Further,

$$x_1 = \frac{x_0}{2} + \frac{A}{2x_0} < \frac{x_0}{2} + \frac{A}{2\sqrt{A}} = \frac{1}{2}\left(x_0 + \sqrt{A}\right).$$

Thus, $\sqrt{A} < x_1 < x_0$. Inductively,

$$\sqrt{A} < x_{m+1} < x_m < \ldots < x_0$$

and $\lim_{m\to\infty} x_m = \sqrt{A}$ by an argument similar to that in Exercise 19(a).
If $0 < x_0 < \sqrt{A}$, then

$$0 < \left(x_0 - \sqrt{A}\right)^2 = x_0^2 - 2x_0\sqrt{A} + A \quad \text{and} \quad 2x_0\sqrt{A} < x_0^2 + A,$$

which leads to

$$\sqrt{A} < \frac{x_0}{2} + \frac{A}{2x_0} = x_1.$$

Thus

$$0 < x_0 < \sqrt{A} < x_{m+1} < x_m < \ldots < x_1,$$

and by the preceding argument, $\lim_{m\to\infty} x_m = \sqrt{A}$.

(b) If $x_0 < 0$, then $\lim_{m\to\infty} x_m = -\sqrt{A}$.

21. Replace the second sentence in the proof with: "Since g satisfies a Lipschitz condition on $[a, b]$ with a Lipschitz constant $L < 1$, we have, for each n,

$$|p_n - p| = |g(p_{n-1}) - g(p)| \leq L|p_{n-1} - p|.\text{"}$$

The rest of the proof is the same, with k replaced by L.

22. Let $\varepsilon = (1 - |g'(p)|)/2$. Since g' is continuous at p, there exists a number $\delta > 0$ such that for $x \in [p-\delta, p+\delta]$, we have $|g'(x) - g'(p)| < \varepsilon$. Thus, $|g'(x)| < |g'(p)| + \varepsilon < 1$ for $x \in [p-\delta, p+\delta]$. By the Mean Value Theorem

$$|g(x) - g(p)| = |g'(c)||x - p| < |x - p|,$$

for $x \in [p - \delta, p + \delta]$. Applying the Fixed-Point Theorem completes the problem.

23. With $g(t) = 501.0625 - 201.0625e^{-0.4t}$ and $p_0 = 5.0,\ p_3 = 6.0028$ is within 0.01 s of the actual time.

24. Since g' is continuous at p and $|g'(p)| > 1$, by letting $\epsilon = |g'(p)| - 1$ there exists a number $\delta > 0$ such that $|g'(x) - g'(p)| < |g'(p)| - 1$ whenever $0 < |x - p| < \delta$. Hence, for any x satisfying $0 < |x - p| < \delta$, we have

$$|g'(x)| \geq |g'(p)| - |g'(x) - g'(p)| > |g'(p)| - (|g'(p)| - 1) = 1.$$

If p_0 is chosen so that $0 < |p - p_0| < \delta$, we have by the Mean Value Theorem that

$$|p_1 - p| = |g(p_0) - g(p)| = |g'(\xi)||p_0 - p|,$$

for some ξ between p_0 and p. Thus, $0 < |p - \xi| < \delta$ so $|p_1 - p| = |g'(\xi)||p_0 - p| > |p_0 - p|$.