1. (a) $\frac{1}{1} + \frac{1}{4} \ldots + \frac{1}{100} = 1.53;\quad \frac{1}{100} + \frac{1}{81} + \ldots + \frac{1}{1} = 1.54.$

   The actual value is 1.549. Significant round-off error occurs much earlier in the first method.

   (b) The following algorithm will sum the series $\sum_{i=1}^{N} x_i$ in the reverse order.

   INPUT $N; x_1, x_2, \ldots, x_N$
   OUTPUT SUM

   STEP 1 Set $SUM = 0$
   STEP 2 For $j = 1, \ldots, N$ set $i = N - j + 1$
   $SUM = SUM + x_i$
   STEP 3 OUTPUT(SUM);
   STOP.

2. We have

	Approximation	Absolute Error	Relative Error
(a)	2.715	3.282×10^{-3}	1.207×10^{-3}
(b)	2.716	2.282×10^{-3}	8.394×10^{-4}
(c)	2.716	2.282×10^{-3}	8.394×10^{-4}
(d)	2.718	2.818×10^{-4}	1.037×10^{-4}

3. (a) 2000 terms

   (b) 20,000,000,000 terms

4. 4 terms

5. 3 terms

6. (a) $O\left(\frac{1}{n}\right)$

   (b) $O\left(\frac{1}{n^2}\right)$

   (c) $O\left(\frac{1}{n^2}\right)$

   (d) $O\left(\frac{1}{n}\right)$

7. The rates of convergence are:

   (a) $O(h^2)$

   (b) $O(h)$

   (c) $O(h^2)$

   (d) $O(h)$

8. (a) $n(n+1)/2$ multiplications; $(n+2)(n-1)/2$ additions.

(b) $\sum_{i=1}^{n} a_i \left(\sum_{j=1}^{i} b_j \right)$ requires n multiplications; $(n+2)(n-1)/2$ additions.

9. The following algorithm computes $P(x_0)$ using nested arithmetic.

INPUT $n, a_0, a_1, \ldots, a_n, x_0$
OUTPUT $y = P(x_0)$

STEP 1 Set $y = a_n$.
STEP 2 For $i = n-1, n-2, \ldots, 0$ set $y = x_0 y + a_i$.
STEP 3 OUTPUT (y);
STOP.

10. The following algorithm uses the most effective formula for computing the roots of a quadratic equation.

INPUT A, B, C.
OUTPUT x_1, x_2.

STEP 1 If $A = 0$ then
if $B = 0$ then OUTPUT ('NO SOLUTIONS');
STOP.
else set $x_1 = -C/B$;
OUTPUT ('ONE SOLUTION',x_1);
STOP.

STEP 2 Set $D = B^2 - 4AC$.

STEP 3 If $D = 0$ then set $x_1 = -B/(2A)$;
OUTPUT ('MULTIPLE ROOTS', x_1);
STOP.

STEP 4 If $D < 0$ then set
$b = \sqrt{-D}/(2A)$;
$a = -B/(2A)$;
OUTPUT ('COMPLEX CONJUGATE ROOTS');
$x_1 = a + bi$;
$x_2 = a - bi$;
OUTPUT (x_1, x_2);
STOP.

STEP 5 If $B \geq 0$ then set
$d = B + \sqrt{D}$;
$x_1 = -2C/d$;
$x_2 = -d/(2A)$
else set
$d = -B + \sqrt{D}$;
$x_1 = d/(2A)$;
$x_2 = 2C/d$.

STEP 6 OUTPUT (x_1, x_2);
STOP.

11. The following algorithm produces the product $P = (x - x_0), \ldots, (x - x_n)$.

INPUT $n, x_0, x_1, \cdots, x_n, x$
OUTPUT P.

STEP 1 Set $P = x - x_0$;
$i = 1$.

STEP 2 While $P \neq 0$ and $i \leq n$ set
$P = P \cdot (x - x_i)$;
$i = i + 1$

STEP 3 OUTPUT (P);
STOP.

12. The following algorithm determines the number of terms needed to satisfy a given tolerance.

INPUT number x, tolerance TOL, maximum number of iterations M.
OUTPUT number N of terms or a message of failure.

STEP 1 Set $SUM = (1-2x)/\left(1-x+x^2\right)$;
$S = (1+2x)/\left(1+x+x^2\right)$;
$N = 2$.

STEP 2 While $N \le M$ do Steps 3–5.

STEP 3 Set $j = 2^{N-1}$;
$y = x^j$
$t_1 = \dfrac{jy}{x}(1-2y)$;
$t_2 = y(y-1)+1$;
$SUM = SUM + \dfrac{t_1}{t_2}$.

STEP 4 If $|SUM - S| < TOL$ then
OUTPUT (N);
STOP.

STEP 5 Set $N = N+1$.

STEP 6 OUTPUT('Method failed');
STOP.

When $TOL = 10^{-6}$, we need to have $N \ge 4$.

13. (a) If $|\alpha_n - \alpha|/(1/n^p) \le K$, then

$$|\alpha_n - \alpha| \le K(1/n^p) \le K(1/n^q) \quad \text{since} \quad 0 < q < p.$$

Thus

$$|\alpha_n - \alpha|/(1/n^p) \le K \quad \text{and} \quad \{\alpha_n\}_{n=1}^{\infty} \to \alpha$$

with rate of convergence $O(1/n^p)$.

(b)

n	$1/n$	$1/n^2$	$1/n^3$	$1/n^5$
5	0.2	0.04	0.008	0.0016
10	0.1	0.01	0.001	0.0001
50	0.02	0.0004	8×10^{-6}	1.6×10^{-7}
100	0.01	10^{-4}	10^{-6}	10^{-8}

The most rapid convergence rate is $O(1/n^4)$.

14. (a) If $F(h) = L + O(h^p)$, there is a constant $k > 0$ such that

$$|F(h) - L| \le kh^p,$$

for sufficiently small $h > 0$. If $0 < q < p$ and $0 < h < 1$, then $h^q > h^p$. Thus, $kh^p < kh^q$, so

$$|F(h) - L| \le kh^q \quad \text{and} \quad F(h) = L + O(h^q).$$

(b) For various powers of h we have the entries in the following table.

h	h^2	h^3	h^4
0.5	0.25	0.125	0.0625
0.1	0.01	0.001	0.0001
0.01	0.0001	0.00001	10^{-8}
0.001	10^{-6}	10^{-9}	10^{-12}

The most rapid convergence rate is $O(h^4)$.

15. Suppose that for sufficiently small $|x|$ we have positive constants k_1 and k_2 independent of x, for which

$$|F_1(x) - L_1| \le K_1|x|^\alpha \quad \text{and} \quad |F_2(x) - L_2| \le K_2|x|^\beta.$$

Let $c = \max(|c_1|, |c_2|, 1)$, $K = \max(K_1, K_2)$, and $\delta = \max(\alpha, \beta)$.

(a) We have

$$\begin{aligned} |F(x) - c_1L_1 - c_2L_2| &= |c_1(F_1(x) - L_1) + c_2(F_2(x) - L_2)| \\ &\le |c_1|K_1|x|^\alpha + |c_2|K_2|x|^\beta \le cK[|x|^\alpha + |x|^\beta] \\ &\le cK|x|^\gamma[1 + |x|^{\delta-\gamma}] \le \tilde{K}|x|^\gamma, \end{aligned}$$

for sufficiently small $|x|$ and some constant $\tilde{K}$. Thus, $F(x) = c_1L_1 + c_2L_2 + O(x^\gamma)$.

(b) We have

$$\begin{aligned} |G(x) - L_1 - L_2| &= |F_1(c_1x) + F_2(c_2x) - L_1 - L_2| \\ &\le K_1|c_1x|^\alpha + K_2|c_2x|^\beta \le Kc^\delta[|x|^\alpha + |x|^\beta] \\ &\le Kc^\delta|x|^\gamma[1 + |x|^{\delta-\gamma}] \le \tilde{K}|x|^\gamma, \end{aligned}$$

for sufficiently small $|x|$ and some constant $\tilde{K}$. Thus, $G(x) = L_1 + L_2 + O(x^\gamma)$.

16. Since

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = x \quad \text{and} \quad x_{n+1} = 1 + \frac{1}{x_n},$$

we have

$$x = 1 + \frac{1}{x}, \quad \text{so} \quad x^2 - x - 1 = 0.$$

The quadratic formula implies that

$$x = \frac{1}{2}\left(1+\sqrt{5}\right).$$

This number is called the *golden ratio*. It appears frequently in mathematics and the sciences.

17. (a) To save space we will show the Maple output for each step in one line. Maple would produce this output on separate lines.

$n := 98; f := 1; s := 1$

$$n := 98 \quad f := 1 \quad s := 1$$

for i from 1 *to n do*
$l := f + s; f := s; s := l;$ *od* :

$l := 2 \quad f := 1 \quad s := 2$
$l := 3 \quad f := 2 \quad s := 3$
$\vdots$
$l := 218922995834555169026 \quad f := 135301852344706746049 \quad s := 218922995834555169026$
$l := 354224848179261915075$

(b) $F100 := \dfrac{1}{sqrt(5)}\left(\left(\dfrac{(1+sqrt(5))}{2}\right)^{100} - \left(\dfrac{1-sqrt(5)}{2}\right)^{100}\right)$

$$F100 := \frac{1}{\sqrt{5}}\left(\left(\frac{1}{2}+\frac{1}{2}\sqrt{5}\right)^{100} - \left(\frac{1}{2}-\frac{1}{2}\sqrt{5}\right)^{100}\right)$$

evalf($F100$)

$$0.3542248538 \times 10^{21}$$

(c) The result in part (a) is computed using exact integer arithmetic, and the result in part (b) is computed using ten-digit rounding arithmetic.

(d) The result in part (a) required traversing a loop 98 times.

(e) The result is the same as the result in part (a).

18. (a) $n = 50$

(b) $n = 500$