1. For each part, $f \in C[a,b]$ on the given interval. Since $f(a)$ and $f(b)$ are of opposite sign, the Intermediate Value Theorem implies that a number $c$ exists with $f(c) = 0$.

2. (a) $[0,1]$
   (b) $[0,1]$, $[4,5]$, $[-1,0]$
   (c) $[-2,-2/3]$, $[0,1]$, $[2,4]$
   (d) $[-3,-2]$, $[-1,-0.5]$, and $[-0.5,0]$

3. For each part, $f \in C[a,b]$, $f'$ exists on $(a,b)$ and $f(a) = f(b) = 0$. Rolle's Theorem implies that a number $c$ exists in $(a,b)$ with $f'(c) = 0$. For part (d), we can use $[a,b] = [-1,0]$ or $[a,b] = [0,2]$.

4. The maximum value for $|f(x)|$ is given below.
   (a) $(2\ln 2)/3 \approx 0.4620981$
   (b) 0.8
   (c) 5.164000
   (d) 1.582572

5. For $x < 0$, $f(x) < 2x + k < 0$, provided that $x < -\frac{1}{2}k$. Similarly, for $x > 0$, $f(x) > 2x + k > 0$, provided that $x > -\frac{1}{2}k$. By Theorem 1.11, there exists a number $c$ with $f(c) = 0$. If $f(c) = 0$ and $f(c') = 0$ for some $c' \neq c$, then by Theorem 1.7, there exists a number $p$ between $c$ and $c'$ with $f'(p) = 0$. However, $f'(x) = 3x^2 + 2 > 0$ for all $x$.

6. Suppose $p$ and $q$ are in $[a,b]$ with $p \neq q$ and $f(p) = f(q) = 0$. By the Mean Value Theorem, there exists $\xi \in (a,b)$ with
$$f(p) - f(q) = f'(\xi)(p - q).$$
But, $f(p) - f(q) = 0$ and $p \neq q$. So $f'(\xi) = 0$, contradicting the hypothesis.

7. (a) $P_2(x) = 0$
   (b) $R_2(0.5) = 0.125$; actual error $= 0.125$
   (c) $P_2(x) = 1 + 3(x-1) + 3(x-1)^2$
   (d) $R_2(0.5) = -0.125$; actual error $= -0.125$

8. $P_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

| $x$ | 0.5 | 0.75 | 1.25 | 1.5 |
|---|---|---|---|---|
| $P_3(x)$ | 1.2265625 | 1.3310547 | 1.5517578 | 1.6796875 |
| $\sqrt{x+1}$ | 1.2247449 | 1.3228757 | 1.5 | 1.5811388 |
| $\lvert\sqrt{x+1} - P_3(x)\rvert$ | 0.0018176 | 0.0081790 | 0.0517578 | 0.0985487 |

9. Since
$$P_2(x) = 1 + x \quad \text{and} \quad R_2(x) = \frac{-2e^{\xi}(\sin\xi + \cos\xi)}{6}x^3$$
for some $\xi$ between $x$ and 0, we have the following:

(a) $P_2(0.5) = 1.5$ and $|f(0.5) - P_2(0.5)| \leq 0.0932$;

(b) $|f(x) - P_2(x)| \leq 1.252$;

(c) $\int_0^1 f(x)\, dx \approx 1.5$;

(d) $|\int_0^1 f(x)\, dx - \int_0^1 P_2(x)\, dx| \leq \int_0^1 |R_2(x)|\, dx \leq 0.313$, and the actual error is 0.122.

10. $P_2(x) = 1.461930 + 0.617884\left(x - \frac{\pi}{6}\right) - 0.844046\left(x - \frac{\pi}{6}\right)^2$ and $R_2(x) = -\frac{1}{3}e^{\xi}(\sin\xi + \cos\xi)\left(x - \frac{\pi}{6}\right)^3$ for some $\xi$ between $x$ and $\frac{\pi}{6}$.

(a) $P_2(0.5) = 1.446879$ and $f(0.5) = 1.446889$. An error bound is $1.01 \times 10^{-5}$, and the actual error is $1.0 \times 10^{-5}$.

(b) $|f(x) - P_2(x)| \leq 0.135372$ on $[0, 1]$

(c) $\int_0^1 P_2(x)\, dx = 1.376542$ and $\int_0^1 f(x)\, dx = 1.378025$

(d) An error bound is $7.403 \times 10^{-3}$, and the actual error is $1.483 \times 10^{-3}$.

11. (a) $P_3(x) = -4 + 6x - x^2 - 4x^3$; $P_3(0.4) = -2.016$

(b) $|R_3(0.4)| \leq 0.05849$; $|f(0.4) - P_3(0.4)| = 0.013365367$

(c) $P_4(x) = -4 + 6x - x^2 - 4x^3$; $P_4(0.4) = -2.016$

(d) $|R_4(0.4)| \leq 0.01366$; $|f(0.4) - P_4(0.4)| = 0.013365367$

12. $P_3(x) = (x-1)^2 - \frac{1}{2}(x-1)^3$

(a) $P_3(0.5) = 0.312500$, $f(0.5) = 0.346574$. An error bound is $0.291\overline{6}$, and the actual error is 0.034074.

(b) $|f(x) - P_3(x)| \leq 0.291\overline{6}$ on $[0.5, 1.5]$

(c) $\int_{0.5}^{1.5} P_3(x)\, dx = 0.08\overline{3}$, $\int_{0.5}^{1.5} (x-1)\ln x\, dx = 0.088020$

(d) An error bound is $0.058\overline{3}$, and the actual error is $4.687 \times 10^{-3}$.

13. $P_4(x) = x + x^3$

(a) $|f(x) - P_4(x)| \leq 0.012405$

(b) $\int_0^{0.4} P_4(x)\, dx = 0.0864$, $\int_0^{0.4} xe^{x^2}\, dx = 0.086755$

(c) $8.27 \times 10^{-4}$

(d) $P_4'(0.2) = 1.12$, $f'(0.2) = 1.124076$. The actual error is $4.076 \times 10^{-3}$.

14. First we need to convert the degree measure for the sine function to radians. We have $180° = \pi$ radians, so $1° = \frac{\pi}{180}$ radians. Since,

$$f(x) = \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x, \quad \text{and} \quad f'''(x) = -\cos x,$$

we have $f(0) = 0$, $f'(0) = 1$, and $f''(0) = 0$.

The approximation $\sin x \approx x$ is given by

$$f(x) \approx P_2(x) = x, \quad \text{and} \quad R_2(x) = -\frac{\cos\xi}{3!}x^3.$$

If we use the bound $|\cos\xi| \leq 1$, then

$$\left|\sin\frac{\pi}{180} - \frac{\pi}{180}\right| = \left|R_2\left(\frac{\pi}{180}\right)\right| = \left|\frac{-\cos\xi}{3!}\left(\frac{\pi}{180}\right)^3\right| \leq 8.86 \times 10^{-7}.$$

15. Since $42° = 7\pi/30$ radians, use $x_0 = \pi/4$. Then

$$\left|R_n\left(\frac{7\pi}{30}\right)\right| \leq \frac{\left(\frac{\pi}{4} - \frac{7\pi}{30}\right)^{n+1}}{(n+1)!} < \frac{(0.053)^{n+1}}{(n+1)!}.$$

For $|R_n(\frac{7\pi}{30})| < 10^{-6}$, it suffices to take $n = 3$. To 7 digits,

$$\cos 42° = 0.7431448 \quad \text{and} \quad P_3(42°) = P_3(\frac{7\pi}{30}) = 0.7431446,$$

so the actual error is $2 \times 10^{-7}$.

16. (a) $P_3(x) = \frac{1}{3}x + \frac{1}{6}x^2 + \frac{23}{648}x^3$

(b) We have

$$f^{(4)}(x) = \frac{-119}{1296}e^{x/2}\sin\frac{x}{3} + \frac{5}{54}e^{x/2}\cos\frac{x}{3},$$

so

$$\left|f^{(4)}(x)\right| \leq \left|f^{(4)}(0.60473891)\right| \leq 0.09787176, \quad \text{for} \quad 0 \leq x \leq 1,$$

and

$$|f(x) - P_3(x)| \leq \frac{|f^{(4)}(\xi)|}{4!}|x|^4 \leq \frac{0.09787176}{24}(1)^4 = 0.004077990.$$

17. (a) $P_3(x) = \ln(3) + \frac{2}{3}(x-1) + \frac{1}{9}(x-1)^2 - \frac{10}{81}(x-1)^3$

(b) $\max_{0\leq x\leq 1}|f(x) - P_3(x)| = |f(0) - P_3(0)| = 0.02663366$

(c) $\tilde{P}_3(x) = \ln(2) + \frac{1}{2}x^2$

(d) $\max_{0\leq x\leq 1}|f(x) - \tilde{P}_3(x)| = |f(1) - \tilde{P}_3(1)| = 0.09453489$

(e) $P_3(0)$ approximates $f(0)$ better than $\tilde{P}_3(1)$ approximates $f(1)$.

18. $P_n(x) = \sum_{k=0}^{n} x^k, \ n \geq 19$

19. $P_n(x) = \displaystyle\sum_{k=0}^{n} \frac{1}{k!}x^k, \ n \geq 7$

20. For $n$ odd, $P_n(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots + \frac{1}{n}(-1)^{(n-1)/2}x^n$. For $n$ even, $P_n(x) = P_{n-1}(x)$.

21. A bound for the maximum error is 0.0026.

22. (a) $P_n^{(k)}(x_0) = f^{(k)}(x_0)$ for $k = 0, 1, \ldots, n$. The shapes of $P_n$ and $f$ are the same at $x_0$.
    (b) $P_2(x) = 3 + 4(x-1) + 3(x-1)^2$.

23. (a) The assumption is that $f(x_i) = 0$ for each $i = 0, 1, \ldots, n$. Applying Rolle's Theorem on each on the intervals $[x_i, x_{i+1}]$ implies that for each $i = 0, 1, \ldots, n-1$ there exists a number $z_i$ with $f'(z_i) = 0$. In addition, we have

$$a \leq x_0 < z_0 < x_1 < z_1 < \cdots < z_{n-1} < x_n \leq b.$$

    (b) Apply the logic in part (a) to the function $g(x) = f'(x)$ with the number of zeros of $g$ in $[a, b]$ reduced by 1. This implies that numbers $w_i$, for $i = 0, 1, \ldots, n-2$ exist with

$$g'(w_i) = f''(w_i) = 0, \quad \text{and} \quad a < z_0 < w_0 < z_1 < w_1 < \cdots w_{n-2} < z_{n-1} < b.$$

    (c) Continuing by induction following the logic in parts (a) and (b) provides $n+1-j$ distinct zeros of $f^{(j)}$ in $[a, b]$.
    (d) The conclusion of the theorem follows from part (c) when $j = n$, for in this case there will be (at least) $(n+1) - n = 1$ zero in $[a, b]$.

24. First observe that for $f(x) = x - \sin x$ we have $f'(x) = 1 - \cos x \geq 0$, because $-1 \leq \cos x \leq 1$ for all values of $x$. Also, the statement clearly holds when $|x| \geq \pi$, because $|\sin x| \leq 1$.

    (a) The observation implies that $f(x)$ is non-decreasing for all values of $x$, and in particular that $f(x) > f(0) = 0$ when $x > 0$. Hence for $x \geq 0$, we have $x \geq \sin x$, and when $0 \leq x \leq \pi$, $|\sin x| = \sin x \leq x = |x|$.
    (b) When $-\pi < x < 0$, we have $\pi \geq -x > 0$. Since $\sin x$ is an odd function, the fact (from part (a)) that $\sin(-x) \leq (-x)$ implies that $|\sin x| = -\sin x \leq -x = |x|$.
    As a consequence, for all real numbers $x$ we have $|\sin x| \leq |x|$.

25. Since $R_2(1) = \frac{1}{6}e^{\xi}$, for some $\xi$ in $(0, 1)$, we have $|E - R_2(1)| = \frac{1}{6}|1 - e^{\xi}| \leq \frac{1}{6}(e - 1)$.

26. (a) Use the series

$$e^{-t^2} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!} \quad \text{to integrate} \quad \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt,$$

and obtain the result.

(b) We have

$$\begin{aligned} \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{1 \cdot 3 \cdots (2k+1)} =& \frac{2}{\sqrt{\pi}} \left[ 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^7 + \frac{1}{24}x^8 + \cdots \right] \\ & \cdot \left[ x + \frac{2}{3}x^3 + \frac{4}{15}x^5 + \frac{8}{105}x^7 + \frac{16}{945}x^9 + \cdots \right] \\ =& \frac{2}{\sqrt{\pi}} \left[ x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \frac{1}{216}x^9 + \cdots \right] = \text{erf}\,(x) \end{aligned}$$

(c) 0.8427008

(d) 0.8427069

(e) The series in part (a) is alternating, so for any positive integer $n$ and positive $x$ we have the bound

$$\left| \text{erf}(x) - \frac{2}{\sqrt{\pi}} \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)k!} \right| < \frac{x^{2n+3}}{(2n+3)(n+1)!}\ .$$

We have no such bound for the positive term series in part (b).

27. (a) Let $x_0$ be any number in $[a, b]$. Given $\epsilon > 0$, let $\delta = \epsilon/L$. If $|x - x_0| < \delta$ and $a \le x \le b$, then $|f(x) - f(x_0)| \le L|x - x_0| < \epsilon$.

(b) Using the Mean Value Theorem, we have

$$|f(x_2) - f(x_1)| = |f'(\xi)||x_2 - x_1|,$$

for some $\xi$ between $x_1$ and $x_2$, so

$$|f(x_2) - f(x_1)| \le L|x_2 - x_1|.$$

(c) One example is $f(x) = x^{1/3}$ on $[0, 1]$.

28. (a) The number $\frac{1}{2}(f(x_1)+f(x_2))$ is the average of $f(x_1)$ and $f(x_2)$, so it lies between these two values of $f$. By the Intermediate Value Theorem 1.11 there exist a number $\xi$ between $x_1$ and $x_2$ with

$$f(\xi) = \frac{1}{2}(f(x_1)+f(x_2)) = \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2).$$

(b) Let $m = \min\{f(x_1), f(x_2)\}$ and $M = \max\{f(x_1), f(x_2)\}$. Then $m \le f(x_1) \le M$ and $m \le f(x_2) \le M$, so

$$c_1 m \le c_1 f(x_1) \le c_1 M \quad \text{and} \quad c_2 m \le c_2 f(x_2) \le c_2 M.$$

Thus

$$(c_1+c_2)m \le c_1 f(x_1) + c_2 f(x_2) \le (c_1+c_2)M$$

and

$$m \le \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1+c_2} \le M.$$

By the Intermediate Value Theorem 1.11 applied to the interval with endpoints $x_1$ and $x_2$, there exists a number $\xi$ between $x_1$ and $x_2$ for which

$$f(\xi) = \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1+c_2}.$$

(c) Let $f(x) = x^2+1$, $x_1 = 0$, $x_2 = 1$, $c_1 = 2$, and $c_2 = -1$. Then for all values of $x$,

$$f(x) > 0 \quad \text{but} \quad \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1+c_2} = \frac{2(1)-1(2)}{2-1} = 0.$$

29. (a) Since $f$ is continuous at $p$ and $f(p) \neq 0$, there exists a $\delta > 0$ with

$$|f(x) - f(p)| < \frac{|f(p)|}{2},$$

for $|x - p| < \delta$ and $a < x < b$. We restrict $\delta$ so that $[p-\delta, p+\delta]$ is a subset of $[a, b]$. Thus, for $x \in [p-\delta, p+\delta]$, we have $x \in [a, b]$. So

$$-\frac{|f(p)|}{2} < f(x) - f(p) < \frac{|f(p)|}{2} \quad \text{and} \quad f(p) - \frac{|f(p)|}{2} < f(x) < f(p) + \frac{|f(p)|}{2}.$$

If $f(p) > 0$, then

$$f(p) - \frac{|f(p)|}{2} = \frac{f(p)}{2} > 0, \quad \text{so} \quad f(x) > f(p) - \frac{|f(p)|}{2} > 0.$$

If $f(p) < 0$, then $|f(p)| = -f(p)$, and

$$f(x) < f(p) + \frac{|f(p)|}{2} = f(p) - \frac{f(p)}{2} = \frac{f(p)}{2} < 0.$$

In either case, $f(x) \neq 0$, for $x \in [p-\delta, p+\delta]$.

(b) Since $f$ is continuous at $p$ and $f(p) = 0$, there exists a $\delta > 0$ with

$$|f(x) - f(p)| < k, \quad \text{for} \quad |x - p| < \delta \quad \text{and} \quad a < x < b.$$

We restrict $\delta$ so that $[p-\delta, p+\delta]$ is a subset of $[a, b]$. Thus, for $x \in [p-\delta, p+\delta]$, we have

$$|f(x)| = |f(x) - f(p)| < k.$$