Homework Assignment for Week 15

1. Reading instruction for section 7.3 and 7.4:

   Study understand meanings of Jacobi, Gauss-Siedel and SOR methods. Learn how to obtain the corresponding T_j, T_g and T_ω and the relation between convergence of the iterations and the spectral properties of the corresponding T's.

   Skip Theorem 7.22, Theorem 7.25 (the proof are given in Section 7.3: Problem 17 and Section 7.4: Problem 12 if you are interested), Theorem 7.26.

2. Section 7.3: Problems 9(a,c), 10(a,c), 14 (see the slides for the Gauss-Seidel part for reference).

3. Section 7.4: Problems 7, 9.

4. Let A be the matrix resulted from discretizing

$$\begin{array}{ll} (\partial_x^2 + \partial_y^2)u(x,y) = f(x,y), & (x,y) \in (0,1)^2 \\ u = 0, \quad \text{on the boundary of } (0,1)^2 & \end{array} \tag{1}$$

   with uniformly spaced grids $0 = x_0 < x_1 < \cdots < x_N = 1$, $0 = y_0 < y_1 < \cdots < y_N = 1$, $x_i - x_{i-1} = y_j - y_{j-1} = h = 1/N$, using second order centered finite difference method. Recall that for this A, the operation count for LU decomposition is N^4, and N^3 for forward and backward substitution.

   (a) Give the operation count for one iteration of Jacobi, Gauss Seidel and SOR on this A, respectively.

   (b) It is a fact (the proof is beyond this course) that **for this** A, $\rho(T_j) \approx 1 - \frac{\pi^2}{2}h^2$, $\rho(T_g) \approx 1 - \pi^2 h^2 \approx \rho(T_j)^2$. Note that both numbers are very close to 1. Take this fact for granted, estimate the number of iterations Jacobi and Gauss-Siedel take to reach $\|e^{(k)}\| = h^2$, respectively (assuming $\|e^{(0)}\| = 1$) where $e^{(k)} = u^{(k)} - u_e$. Then estimate the total number of multiplications/divisions needed for Jacobi and Gauss-Siedel, respectively. You will need that $\log(1+x) \approx x$ for $|x| << 1$.

   (c) It is another fact (the proof is also beyond this course) that with the optimal $\omega = \omega^* \approx 2 - 2\pi h$, we will have $\rho(T_{\omega^*}) \approx 1 - 2\pi h$ for SOR (for this A). Take this fact for granted again, estimate the number of iterations and total multiplications/divisions the optimal SOR takes to reach $\|e^{(k)}\| = h^2$ with $\|e^{(0)}\| = 1$.

   (d) The facts that $\rho(T_j) \approx 1 - C_1 h^2$, $\rho(T_g) \approx 1 - C_2 h^2$ and $\rho(T_{\omega^*}) \approx 1 - C_3 h$ remain valid in the 3D version of (1), with different constants C_i. Repeat the above problems for the 3D case.

   The purpose of this problem is to demonstrate that, beating Gauss Elimination/LU decomposition with iterative methods is possible, but sometimes non-trivial.

5. Derive the Jacobi version of SOR. Express $T_{\omega,j}$ in terms of T_j.