Quiz 01

1. Store a floating point number in the range

$$\pm 1.a_1 a_2 \cdots a_s \times 2^e$$

with $s = 33$, $a_i \in \{0,1\}$, $-511 \le e \le 512$

Since $512 + 511 + 1 = 1024 = 2^{10} \Rightarrow$ we need 10 bits to store "e"

$$\underset{\displaystyle 1}{\overset{\pm}{\downarrow}} + \underset{\displaystyle 10}{\overset{e}{\downarrow}} + \underset{\displaystyle 33}{\overset{a_1 a_2 \cdots a_{33}}{\downarrow}} = 44 \text{ (bits)}$$

2. Since $(100002)^{\frac{1}{3}}$, $(100001)^{\frac{1}{3}}$ is very close, we have to use some other way to evaluate the value $(a^3 - b^3 = (a-b)(a^2+ab+b^2))$

$$(100002)^{\frac{1}{3}} - (100001)^{\frac{1}{3}} = \frac{100002 - 100001}{(100002)^{\frac{2}{3}} + (100002)^{\frac{1}{3}}(100001)^{\frac{1}{3}} + (100001)^{\frac{2}{3}}}$$

$$= 1.547180806108708 \times 10^{-4}$$

3. (a)

$\lim_{h \to 0} \frac{\sin h}{h} = 1$ and $\sin h = h - \frac{h^3}{3!}\cos(\xi_1)$ where ξ_1 is between 0 and h

$$\left| \frac{\sin h}{h} - 1 \right| = \left| \frac{h^2}{6}\cos(\xi_1) \right| \le \frac{1}{6}|h^2| \Rightarrow \frac{\sin h}{h} = 1 + O(h^2)$$

(b) $\lim_{n \to \infty} (\ln(n+1) - \ln n) = \lim_{n \to \infty} \ln \frac{n+1}{n} = 0$

and $\ln\left(1 + \frac{1}{n}\right) = 0 + \frac{1}{1+\xi_2} \cdot \frac{1}{n}$ where ξ_2 is between 0 and $\frac{1}{n}$

$$\left| (\ln(n+1) - \ln n) - 0 \right| = \left| \frac{1}{1+\xi_2} \cdot \frac{1}{n} \right| \le \left| \frac{1}{n} \right| \Rightarrow \ln(n+1) - \ln n = 0 + O\left(\frac{1}{n}\right)$$

4. $P_0 = 1$, $P_1 = \frac{1}{3}$, $P_n = \frac{10}{3} P_{n-1} - P_{n-2}$, this algorithm is unstable

Let P_n^e be the exact solution and P_n^h be the numerical solution

$$\Rightarrow \begin{cases} P_0^e = 1, \quad P_1^e = \frac{1}{3} \\ P_n^e = \frac{10}{3} P_{n-1}^e - P_{n-2}^e \end{cases} \quad \text{and} \quad \begin{cases} P_0^h = 1, \quad P_1^h = \frac{1}{3}(1+\delta) \quad \text{for some } \delta > 0 \\ P_n^h = \frac{10}{3} P_{n-1}^h - P_{n-2}^h \end{cases}$$

Let $e_n = P_n^h - P_n^e \Rightarrow \begin{cases} e_0 = 0, \quad e_1 = \frac{1}{3}\delta \\ e_n = \frac{10}{3} e_{n-1} - e_{n-2} \end{cases} \Rightarrow e_n = c_1 (\frac{1}{3})^n + c_2 3^n$

$$\Rightarrow \begin{cases} c_1 + c_2 = e_0 = 0 \\ \frac{1}{3} c_1 + 3 c_2 = e_1 = \frac{1}{3}\delta > 0 \end{cases} \Rightarrow c_2 = \frac{\delta}{8} > 0$$

Hence the error grows exponentially in n

5.

Numerical solution of a root of $x^3 + 3x - 3 = 0$ with absolute error less than 2^{-20} is : $8.177309036254883 \times 10^{-1}$

Since $a_0 = \frac{1}{2}$, $b_0 = 1$, let $c_n = \frac{a_{n-1} + b_{n-1}}{2}$, $n = 1, 2, \cdots$

and $\tilde{x}$ be the root of $x^3 + 3x - 3 = 0$

then $|c_n - \tilde{x}| \le \frac{|b_0 - a_0|}{2^n} = 2^{-n-1}$

Find n s.t. $2^{-n-1} \le 2^{-20} \Rightarrow -n-1 \le -20 \Rightarrow n \ge 19$

Hence $|c_n - \tilde{x}| \le 2^{-n-1} \le 2^{-20}$, $\forall n \ge 19$