$\#_1$

$$[A \mid b] = \left[\begin{array}{ccccc|c} a_{11} & \cdots & a_{1,N+1} & & O & b_1 \\ \vdots & \ddots & & \ddots & & b_2 \\ a_{N+1,1} & & \ddots & & \ddots & \vdots \\ & \ddots & & \ddots & a_{N^2-N,N^2} & \vdots \\ O & & \ddots & & \vdots & \vdots \\ & & a_{N^2,N^2-N} & \cdots & a_{N^2,N^2} & b_{N^2} \end{array}\right] = [A^{(1)} \mid b^{(1)}]$$

① use $E_i^{(i)}$ to eliminate x_i from the equation $E_{i+1}^{(i)} \cdots E_{i+N}^{(i)}$

need N multiplications to compute m_{ij}, $i+1 \le j \le i+N$

and

$$\left.\begin{array}{lll} (E_{i+1}^{(i)} - m_{i,i+1}E_i^{(i)}) \to E_{i+1}^{(i+1)} & \text{need} & N+1 \\ (E_{i+2}^{(i)} - m_{i,i+2}E_i^{(i)}) \to E_{i+2}^{(i+1)} & \text{need} & N+1 \\ \vdots & & \vdots \\ (E_{i+N}^{(i)} - m_{i,i+N}E_i^{(i)}) \to E_{i+N}^{(i+1)} & \text{need} & N+1 \end{array}\right\} \text{total } N(N+1)$$

for $1 \le i \le N^2-N$

So. $$[A^{(1)} \mid b^{(1)}] \to [A^{(N^2-N+1)} \mid b^{(N^2-N+1)}] = \left[\begin{array}{cccc|c} x & \cdots & x & O & b_1^{(N^2-N+1)} \\ O & x & \cdots & x & \vdots \\ O & & \ddots & \vdots & \vdots \\ & & x & \cdots\ x & b_{N^2}^{(N^2-N+1)} \end{array}\right]$$

we need $(N^2-N)(N)(N+1)$ multiplications

②

$$\left[A^{(N^2-N+1)} \middle| b^{(N^2-N+1)}\right] = \left[\begin{array}{ccc|c} x & \cdots & x & \\ & x & \cdots x & b^{(N^2-N+1)} \\ & x & \cdots x & \end{array}\right] \begin{array}{l} \leftarrow E_{N^2-N+1}^{(N^2-N+1)} \\ \\ \leftarrow E_{N^2}^{(N^2-N+1)} \end{array}$$

we focus on nxn submatrix that every entries are non zero

$\Rightarrow$ we need $\frac{2N^3+3N^2-5N}{6}$ multip. such that

$$\left[A^{(N^2-N+1)} \middle| b^{(N^2-N+1)}\right] \to \left[\begin{array}{ccc|c} x & \cdots x & 0 & b^{(N^2)} \\ 0 & \ddots & x & \\ & & x & \end{array}\right] = \left[A^{(N^2)} \middle| b^{(N^2)}\right]$$

③ $\left[A^{(N^2)} \middle| b^{(N^2)}\right] \to \left[A^{(2N^2)} \middle| b^{(2N^2)}\right]$

where $A^{(2N^2)}$ is diagonal matrix

we need $(2N)(N^2-N) + 2\sum_{j=1}^{N-1}(N-j)$

④ need N^2 to compute solution $x_i = \frac{b^{(2N^2)}}{a_{ii}^{(2N^2)}}$

by ①②③④, total $= (N^2-N)(N)(N+1) + \frac{2N^3+3N^2-5N}{6}$

$$+ (2N)(N^2-N) + 2\sum_{j=1}^{N-1}(N-j) + N^2$$

$$= N^4 + \cdots\cdots$$