Quiz 05

1. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$

find (a_1, a_2) that minimizes $\left\| b - A\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \right\|$

Let $f(a_1, a_2) = \left\| b - A\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \right\|^2$

$$= (a_1 + 2a_2)^2 + (2a_1 - a_2 - 1)^2 + (3a_1 - a_2 - 5)^2$$

$$\frac{\partial f(a_1, a_2)}{\partial a_1} = 14a_1 - 3a_2 - 17 \overset{\text{hope}}{=} 0$$

$$\frac{\partial f(a_1, a_2)}{\partial a_2} = -3a_1 + 6a_2 + 6 \overset{\text{hope}}{=} 0$$

$$\Rightarrow \begin{pmatrix} 14 & -3 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -17 \\ 6 \end{pmatrix}, \quad \text{"} A^t A\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = A^t b \text{"}$$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 28/25 \\ -11/25 \end{pmatrix}$$

2. from textbook P.75

Definition 2.6

$$\lim_{n\to\infty} \frac{|a_{n+1}-1|}{|a_n-1|^{\alpha}} = \lim_{n\to\infty} \frac{\left|\frac{1}{(n+1)^2}\right|}{\left|\frac{1}{n^2}\right|^{\alpha}} \overset{\text{if } \alpha=1}{=\!=} 1$$

So, $\{a_n\}_{n=0}^{\infty}$ converges to 1 of order 1,

with asymptotic error constant 1

example: let $b_n = 10^{-2^n}$

then $$\frac{|b_{n+1}|}{|b_n|^{\alpha}} = \frac{10^{-2^{n+1}}}{10^{-2^n \cdot \alpha}} = 10^{2^n \cdot \alpha - 2^{n+1}}$$

if $\alpha < 2$ $\lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} = 0$

if $\alpha = 2$ $\lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} = 1$

※

(本題為 2.4，超出範圍送分。
答對者另加 10分)

3, $f(x) = x - 2\cos x$

Let $g(x) = 2\cos x$, if x^* be the fixed point of $g(x)$

then $x^* = g(x^*) = 2\cos x^*$

$\Rightarrow \quad f(x^*) = x^* - 2\cos x^* = 0$

$\Rightarrow \quad x^*$ is solution of $f(x) = 0$

But $g'(x) \approx 1.71$ near $x = x^*$ $\quad (|g'(x)| > 1)$

So, we must change the iteration formula

Let $h(x) = \alpha x + (1-\alpha) g(x)$

$h'(x) = \alpha + (1-\alpha) g'(x)$

if we want $|h'(x)| < 1$ near $x = x^*$

$\Rightarrow \quad |\alpha + (1-\alpha) g'(x^*)| < 1$

$\Rightarrow \quad |1.71 - 0.71\alpha| < 1$

$\Rightarrow \quad 1 < \alpha < \frac{2.71}{0.71} = 3.8169$

$\left(\begin{array}{l}\text{Best } \alpha: \\ 1.71 - 0.71\alpha \approx 0 \\ \alpha \doteqdot 2.4085\end{array}\right)$

choose $\alpha = 2.4085$, then $|h'(x)| \approx 0 < 1$ near $x = x^*$

Note: The fixed point of h is also a fixed point of g

Hence, The iteration formula is:

$$x_{n+1} = (2.4085)(x_n) - (1.4085)\cos(x_n)$$

#

4. Pseudo-code for Newton's method:

```
p0 = 1;
M = 50;   % maximum number of iterations
tol = 10^-10;   % tolerance
k = 1;
while ((k < M) & (tol < |f(p0)|))
    p1 = p0 - (p0 - 2cos(p0)) / (1 + 2sin(p0));   % (x_{n+1} = x_n - f(x_n)/f'(x_n))
    k = k+1;
    p0 = p1;
end
OUTPUT (k, p0)
```

Pseudo-code for secant method:

```
p0 = 0; p1 = 1; q0 = f(p0); q1 = f(p1);
M = 50; tol = 10^-10;
k = 2
while ((k < M) & (tol < |f(p1)|))
    p2 = p1 - q1(p1 - p0) / (q1 - q0);
    k = k+1;
    p0 = p1; q0 = q1;
    p1 = p2; q1 = f(p2);
end
OUTPUT (p1)
```

#