1. (a) $u''(x) + u(x) = f(x) \quad , \quad x \in (0,1)$

$u(0) = u(1) = 0$

Let $h = \frac{1}{N}$, $x_j = jh$, $u(x_j) = u_j$

second order finite difference methed :

use $\dfrac{u_{j-1} - 2u_j + u_{j+1}}{h^2} = u''(x_j)$

$$\Rightarrow \underbrace{\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & 0 & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix}}_{\overset{\text{Let}}{=} A} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix} + \begin{bmatrix} \frac{u_0}{h^2} = 0 \\ 0 \\ \vdots \\ 0 \\ \frac{u_N}{h^2} = 0 \end{bmatrix} = \begin{bmatrix} u''(x_1) \\ u''(x_2) \\ \vdots \\ u''(x_{N-1}) \end{bmatrix}$$

$$\Rightarrow (A + I) \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} u''(x_1) + u(x_1) \\ u''(x_2) + u(x_2) \\ \vdots \\ u''(x_{N-1}) + u(x_{N-1}) \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \end{bmatrix} = b$$

(b) Need $(N-2) \cdot 3 + 1 + \underbrace{2 + 2 + \cdots + 2}_{N-2}$

$= 5(N-2) + 1 = 5N - 9$ multip. $\Rightarrow C = 5 , p = 1$

#

2,

$$[B|c] = \left[\begin{array}{cccccccccc|c}
b_{11} & b_{12} & 0 & \cdots & 0 & b_{1,M+1} & & & & & c_1 \\
b_{21} & b_{22} & b_{23} & 0 & \cdots & 0 & b_{2,M+2} & & & & c_2 \\
0 & b_{32} & b_{33} & b_{34} & & & & \ddots & & 0 & c_3 \\
\vdots & 0 & \ddots & \ddots & \ddots & & & & \ddots & & \vdots \\
0 & \vdots & & \ddots & \ddots & \ddots & & & & b_{M^3-M,M^3} & \\
b_{M+1,1} & 0 & & & \ddots & \ddots & \ddots & & & 0 & \\
 & b_{M+2,1} & & & & \ddots & \ddots & \ddots & & \vdots & \\
 & & \ddots & 0 & & & \ddots & \ddots & \ddots & 0 & \\
 & & & \ddots & & & & \ddots & \ddots & b_{M^3-1,M^3} & \\
 & & & & b_{M^3,M^3-M} & 0 & \cdots & 0 & b_{M^3,M^3-1} & b_{M^3,M^3} & c_{M^3}
\end{array}\right]$$

Let $E_j^{(i)}$ denote jth equation " $b_{j,1}x_1 + b_{j,2}x_2 + \cdots + b_{j,M^3}x_{M^3} = c_j$ and the upper index (i) denote the new equation after do "$i-1$" steps elimination. $(B = B^{(1)})$

Step I: use $E_1^{(1)}$ to eliminate x_1 from the equation $E_2^{(1)}$, $E_{M+1}^{(1)}$

we need compute $m_{1,2} = \dfrac{b_{21}^{(1)}}{b_{11}^{(1)}}$, need 1 multip.

$m_{1,M+1} = \dfrac{b_{M+1,1}^{(1)}}{b_{11}^{(1)}}$, need 1 multip.

and $(E_2^{(1)} - m_{12}E_1^{(1)}) \longrightarrow E_2^{(2)}$, need 3 multip.

$(E_{M+1}^{(1)} - m_{1,M+1}E_1^{(1)}) \rightarrow E_{M+1}^{(2)}$, need 3 multip.

Note that, Because we know $b_{21}^{(2)} = b_{21}^{(1)} - \dfrac{b_{21}^{(1)}}{a_{11}^{(1)}} \cdot a_{11}^{(1)} = 0$

so, we don't need to compute it, just let $b_{21}^{(2)} = 0$

After step 1, the new equation $E_2^{(2)}$ add one nonzero entry ($b_{2,M+1}^{(1)} = 0 \longrightarrow b_{2,M+1}^{(2)} = -\frac{b_{21}^{(1)}}{b_{11}^{(1)}} \cdot b_{1,M+1}$)

and the new equation $E_{M+1}^{(2)}$ add one nonzero entry

Step 2: use $E_2^{(2)}$ to elimination x_2 from the equation

$E_3^{(2)}$, $E_{M+1}^{(2)}$, $E_{M+2}^{(2)}$

$\because$ the new equation $E_2^{(2)}$ still has 4 nonzero entries

$\therefore$ we need $3 + 3 \times 4 = 3(4+1)$ multip.

After step 2. the new equation $E_3^{(3)}$ add two nonzero entry ($b_{3,M+1}^{(3)} \neq 0$, $b_{3,M+2}^{(3)}$)

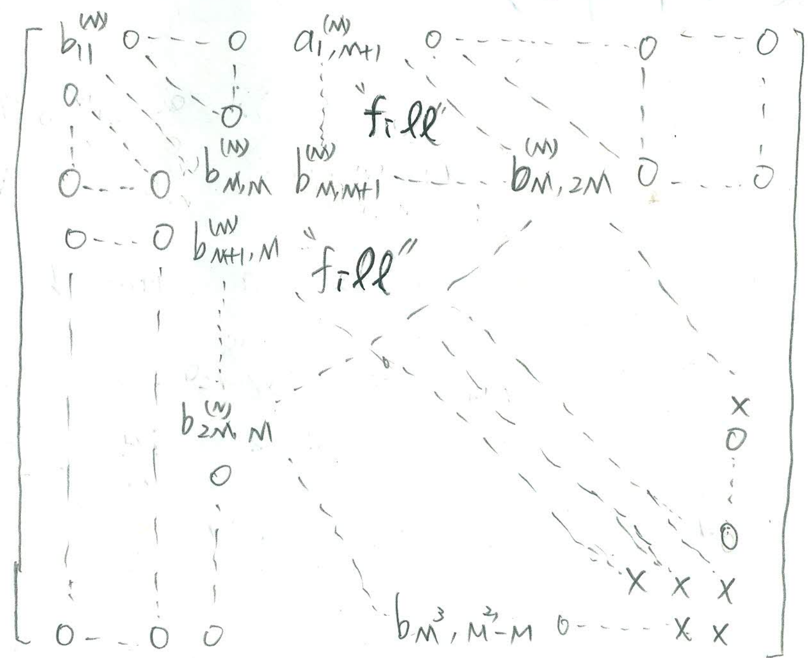
and $E_{M+1}^{(3)}(3) \neq 0$ $E_{M+2}^{(3)}(3) \neq 0$.

($E_j^{(i)}(k)$ denote kth entry in equation $E_j^{(i)}$)

so, the new equation $E_{(3)}^{(3)}$ has 5 nonzero entries.

untill, step M:

$[B^{(N)} \mid C^{(N)}] =$



$\because E_M^{(M)}$ has M+1 nonzero entries, & In column M has M+1 nonzero netries.

Hence: use $E_M^{(M)}$ to eliminate x_M from the equation

$\underbrace{E_{M+1}^{(M)}, E_{M+2}^{(M)}, \cdots E_{2M}^{(M)}}_{M}$, we need $M+(M+1)\cdot M$

$= (M+2)M$ multip.

From "step M" to "step M^3-M", each step need

"$(M+2)(M)$" multip.

So, total need $(M^3-2M+1)(M)(M+2)$

$= M^5 + \cdots$

And the following work "step $M^3-M+1 \sim$ step end"

need no more then CM^4, for some C

thus, the leading order is M^5

$\Rightarrow K=1, \; q=5$

3, ① Gaussian elimination:

$$\begin{array}{c}\begin{pmatrix}-1&2&2&0\\2&2&0&0\\2&-1&1&4\\-2&4&1&5\end{pmatrix}\begin{array}{l}\\(2)\\(2)\\(-2)\end{array}\end{array}\rightarrow\begin{array}{c}\begin{pmatrix}-1&2&2&0\\0&6&4&0\\0&3&5&4\\0&0&-3&5\end{pmatrix}\begin{array}{l}\\\\(-\frac{1}{2})\\\\\end{array}\end{array}\rightarrow\begin{array}{c}\begin{pmatrix}-1&2&2&0\\0&6&4&0\\0&0&3&4\\0&0&-3&5\end{pmatrix}\begin{array}{l}\\\\\\(1)\end{array}\end{array}$$

$$\rightarrow\begin{pmatrix}-1&2&2&0\\0&6&4&0\\0&0&3&4\\0&0&0&9\end{pmatrix}=U_g$$

② Gaussian elimination with partial pivoting:

$$\begin{pmatrix}-1&2&2&0\\2&2&0&0\\2&-1&1&4\\-2&4&1&5\end{pmatrix}\rightarrow\begin{array}{c}\begin{pmatrix}2&2&0&0\\-1&2&2&0\\2&-1&1&4\\-2&4&1&5\end{pmatrix}\begin{array}{l}\\(\frac{1}{2})\\(-1)\\(1)\end{array}\end{array}\rightarrow\begin{pmatrix}2&2&0&0\\0&3&2&0\\0&-3&1&4\\0&6&1&5\end{pmatrix}$$

$$\rightarrow\begin{array}{c}\begin{pmatrix}2&2&0&0\\0&6&1&5\\0&-3&1&4\\0&3&2&0\end{pmatrix}\begin{array}{l}\\\\(\frac{1}{2})\\(-\frac{1}{2})\end{array}\end{array}\rightarrow\begin{array}{c}\begin{pmatrix}2&2&0&0\\0&6&1&5\\0&0&3/2&13/2\\0&0&3/2&-5/2\end{pmatrix}\begin{array}{l}\\\\\\(-1)\end{array}\end{array}\rightarrow\begin{pmatrix}2&2&0&0\\0&6&1&5\\0&0&3/2&13/2\\0&0&0&-9\end{pmatrix}=U_p$$

③ Gaussian elimination with scale pratial piroting

$s_1=2$, $\frac{|a_{11}|}{s_1}=\frac{1}{2}$
$s_2=2$, $\frac{|a_{21}|}{s_2}=1$
$s_3=4$, $\frac{|a_{31}|}{s_3}=\frac{1}{2}$
$s_4=5$, $\frac{|a_{41}|}{s_4}=\frac{2}{5}$

$$\begin{pmatrix}-1&2&2&0\\2&2&0&0\\2&-1&1&4\\-2&4&1&5\end{pmatrix}\rightarrow\begin{array}{c}\begin{pmatrix}2&2&0&0\\-1&2&2&0\\2&-1&1&4\\-2&4&1&5\end{pmatrix}\begin{array}{l}\\(\frac{1}{2})\\(-1)\\(1)\end{array}\end{array}\rightarrow$$

$s_2=2$, $\frac{|a_{22}|}{s_2}=\frac{3}{2}$
$s_3=4$, $\frac{|a_{32}|}{s_3}=\frac{3}{4}$
$s_4=5$, $\frac{|a_{42}|}{s_4}=\frac{5}{2}$

$$\begin{array}{c}\begin{pmatrix}2&2&0&0\\0&3&2&0\\0&-3&1&4\\0&6&1&5\end{pmatrix}\begin{array}{l}\\\\(1)\\(-2)\end{array}\end{array}$$

$\rightarrow$ $s_3=4$, $\frac{|a_{33}|}{s_3}=\frac{3}{4}$
$s_4=5$, $\frac{|a_{43}|}{s_4}=\frac{3}{5}$

$$\begin{array}{c}\begin{pmatrix}2&2&0&0\\0&3&2&1\\0&0&3&4\\0&0&-3&5\end{pmatrix}\begin{array}{l}\\\\\\(1)\end{array}\end{array}\rightarrow\begin{pmatrix}2&2&0&0\\0&3&2&0\\0&0&3&4\\0&0&0&9\end{pmatrix}=U_s$$

4. ① G-E

$$\because \underbrace{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix}}_{E_3}\underbrace{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-\frac{1}{2}&1&0\\0&0&0&1\end{pmatrix}}_{E_2}\underbrace{\begin{pmatrix}1&0&0&0\\2&1&0&0\\2&0&1&0\\-2&0&0&1\end{pmatrix}}_{E_1}A=U_g$$

$$\therefore A = E_1^{-1}E_2^{-1}E_3^{-1}U_1 = \begin{pmatrix}1&0&0&0\\-2&1&0&0\\-2&\frac{1}{2}&1&0\\2&0&-1&1\end{pmatrix}U_1 = L_gU_g$$ ※

② G-E with partial pivoting:

$$\because \underbrace{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-1&1\end{pmatrix}}_{E_3}\underbrace{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&\frac{1}{2}&1&0\\0&-\frac{1}{2}&0&1\end{pmatrix}}_{E_2'}\underbrace{\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}}_{P_2}\underbrace{\begin{pmatrix}1&0&0&0\\\frac{1}{2}&1&0&0\\-1&0&1&0\\1&0&0&1\end{pmatrix}}_{E_1}\underbrace{\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}}_{P_1}A=U_p$$

$$\therefore P_1A = E_1^{-1}P_2^{t}E_2^{-1}E_3^{-1}U_p$$

$$P_2P_1A = P_2E_1^{-1}P_2^{t}E_2^{-1}E_3^{-1}U_p$$

$$\because P_2E_1^{-1}P_2^{t} = \begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\-\frac{1}{2}&1&0&0\\1&0&1&0\\-1&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}$$

$$= \begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\-\frac{1}{2}&0&0&1\\1&0&1&0\\-1&1&0&0\end{pmatrix} = \begin{pmatrix}1&0&0&0\\-1&1&0&0\\1&0&1&0\\-\frac{1}{2}&0&0&1\end{pmatrix} = L_1$$

$$\therefore P_2P_1A = \underbrace{\begin{pmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{pmatrix}}_{P}A = L_1E_2^{-1}E_3^{-1}U_p = \begin{pmatrix}1&0&0&0\\-1&1&0&0\\1&-\frac{1}{2}&1&0\\-\frac{1}{2}&\frac{1}{2}&1&1\end{pmatrix}U_p = L_pU_p$$

4. ③ G-E with scale partial pivoting :

$$\because \underbrace{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix}}_{E_3}\underbrace{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&-2&0&1\end{pmatrix}}_{E_2}\underbrace{\begin{pmatrix}1&0&0&0\\\frac12&1&0&0\\-1&0&1&0\\1&0&0&1\end{pmatrix}}_{E_1}\underbrace{\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}}_{P_1}A = U_s$$

$$\therefore P_1 A = E_1^{-1}E_2^{-1}E_3^{-1}U_s$$

$$= \underline{\begin{pmatrix}1&0&0&0\\-\frac12&1&0&0\\1&-1&1&0\\-1&2&-1&1\end{pmatrix}} U_s = \underline{\underline{L_s U_s}}$$

$$P = P_1 = \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \quad \#$$