Quiz 02

1. $N_2(h) = N_1(\frac{h}{2}) + \dfrac{N(\frac{h}{2}) - N(h)}{4-1} = 2.00456$

$N_2(\frac{h}{2}) = N_1(\frac{h}{4}) + \dfrac{N(\frac{h}{4}) - N(\frac{h}{2})}{4-1} = 2.00069\overline{6}$

$N_3(h) = N_2(\frac{h}{2}) + \dfrac{N_2(\frac{h}{2}) - N_2(h)}{16-1} = 1.9999836\overline{4}$

$O(h^2)$	$O(h^4)$	$O(h^6)$
$N_1(h)$		
$N_1(\frac{h}{2})$	$N_2(h)$	
$N_1(\frac{h}{4})$	$N_2(\frac{h}{2})$	$N_3(h)$

2. $\log_3\left(\dfrac{N(h) - N(\frac{h}{3})}{N(\frac{h}{3}) - N(\frac{h}{9})}\right) = 1.499942525051934$

≈ 1.5 #

3. $\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12} h^2 f''(\mu)$

Absolute error $= \left|\frac{b-a}{12} h^2 f''(\mu)\right| = \left|\frac{2-1}{12} h^2 (\cos x^2)''_{x=\mu}\right|$

$= \frac{1}{12} h^2 \left|-2\sin\mu^2 - 4\mu^2\cos\mu^2\right|$

$\leq \frac{1}{12} h^2 (2 + 16) = \frac{18}{12} h^2 \overset{\text{hope}}{<} 10^{-5}$

Let $h = \frac{1}{n} \Rightarrow n \geq \sqrt{\frac{18}{12} \times 10^5} = 387.29833\ldots$

$\Rightarrow n \geq 388$

code:

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h = 1/388
x = 1:h:2
y = cos(x.^2)
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Ans $= (\frac{h}{2})[y(1) + 2\,\mathrm{sum}(y(2{:}1{:}\mathrm{end}-1)) + y(\mathrm{end})]$

$= -0.443060168169741$

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4. $\displaystyle\int_{-h}^{h} f(x)\,dx \approx \frac{h}{3}\left(f(-h)+4f(0)+f(h)\right)$

5. (i) $f(x)=1$, $\displaystyle\int_{-h}^{h} 1\,dx = 2h = 2h(\alpha_- + \alpha_+)$

$\Rightarrow \alpha_- + \alpha_+ = 1$

(ii) $f(x)=x$, $\displaystyle\int_{-h}^{h} x\,dx = 0 = 2h\left(\alpha_-\cdot\frac{-h}{2} + \alpha_+\cdot\frac{h}{2}\right)$

$\Rightarrow \alpha_+ - \alpha_- = 0$

by (i),(ii) $\Rightarrow \alpha_+ = \alpha_- = \frac{1}{2}$

(iii) $f(x)=x^2$, $\displaystyle\int_{-h}^{h} x^2\,dx = \frac{1}{3}x^3\Big|_{-h}^{h} = \frac{2}{3}h^3$

and $2h(\alpha_- + \alpha_+)\left(\frac{h}{2}\right)^2 = 2h\left(\frac{h^2}{4}\right) = \frac{h^3}{2} \neq \frac{2}{3}h^3$ $(h \neq 0)$

Hence degree $= 1$ #

expand $f(x)$, $f(\frac{h}{2})$, $f(-\frac{h}{2})$ around 0 we have

$$\begin{cases} f(x) = f(0) + f'(0)x + \dfrac{f''(\xi_1(x))}{2!}x^2, & \xi_1(x) \in (0,x) \\ f(\frac{h}{2}) = f(0) + f'(0)\dfrac{h}{2} + \dfrac{f''(\xi_2)}{2!}\left(\dfrac{h}{2}\right)^2, & \xi_2 \in (0,\frac{h}{2}) \\ f(-\frac{h}{2}) = f(0) - f'(0)\dfrac{h}{2} + \dfrac{f''(\xi_3)}{2!}\left(\dfrac{h}{2}\right)^2, & \xi_3 \in (-\frac{h}{2},0) \end{cases}$$

and $\displaystyle\int_{-h}^{h} f(x)\,dx = f(0)\cdot(2h) + \frac{f''(\xi)}{2!}\cdot 2\cdot\frac{h^3}{3}$, for some $\xi \in (-h,h)$

$$\Rightarrow \left|\int_{-h}^{h} f(x)\,dx - 2h\left(\frac{f(\frac{h}{2})+f(-\frac{h}{2})}{2}\right)\right| = \left|\frac{f''(\xi)}{3}h^3 - \frac{f''(\xi_2)}{8}h^3 - \frac{f''(\xi_3)}{8}h^3\right|$$

$$\leq \left(\frac{1}{3}+\frac{1}{8}+\frac{1}{8}\right)h^3 \max\{|f''(\xi)|, |f''(\xi_2)|, |f''(\xi_3)|\}$$

$$\leq \frac{7}{12}h^3 \max_{x\in(-h,h)}\{|f''(x)|\}$$

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