Quiz 01

1. Store a floating point number in the range

$$\pm 1.a_1 a_2 \cdots a_s \times 2^e$$

with $s=11$, $a_j \in \{0,1\}$, $-6 \le e \le 7$

(1) Since $6+7+1=14 < 2^4 \Rightarrow$ we need 4 bits to store "e",

$$\underset{\pm}{1} + \underset{e}{4} + \underset{a_1 a_2 \cdots a_s}{11} = \underline{\underline{16}} \text{ (bits)}$$

(2) "$(-1)^s 2^{c-7}(1+f)$", $\because 0 \le c \le 15$ $\therefore -7 \le c-7 \le 8$

(選"7"可讓頭尾保留1位)

$$-0.625 = -1.01 \times 2^{-1}$$

$$= \underline{1}\ \underline{0110}\ \underline{01000000000}$$ #

2.

look textbook P23, Example 5

3.

① $\left|\frac{\pi^2}{6} - \sum_{n=1}^{N} \frac{1}{n^2}\right| = \left|\sum_{n=N+1}^{\infty} \frac{1}{n^2}\right| = \sum_{n=N+1}^{\infty} \frac{1}{n^2} \le \int_N^\infty \frac{1}{x^2}\,dx = \frac{1}{N}$

② $\sum_{n=N+1}^{\infty} \frac{1}{n^2} \ge \int_{N+1}^{\infty} \frac{1}{x^2}\,dx = \frac{1}{N+1}$

$\Rightarrow \frac{1}{N+1} \le \left|\frac{\pi^2}{6} - \sum_{n=1}^{N} \frac{1}{n^2}\right| \le \frac{1}{N}$, $O\left(\frac{1}{N}\right)$

#

4.

(1) $P_n = \frac{10}{3}P_{n-1} - P_{n-2}$, $x^2 - \frac{10}{3}x + 1 = 0 \Rightarrow x = 3$ or $x = 1/3$

$\Rightarrow P_n = C_1(\frac{1}{3})^n + C_2(3)^n$, $\because \begin{cases} P_0 = C_1 + C_2 = 1 \\ P_1 = \frac{1}{3}C_1 + 3C_2 = 1/3 \end{cases} \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = 0 \end{cases} \Rightarrow P_n = (\frac{1}{3})^n$ ※

(2) unstable, Let P_n^e be the exact solution and P_n^h be the numerical solution

$$\Rightarrow \begin{cases} P_0^e = 1 \\ P_1^e = \frac{1}{3} \\ P_n^e = \frac{10}{3}P_{n-1}^e - P_{n-2}^e \end{cases} \text{ and } \begin{cases} P_0^h = 1 \\ P_1^h = \frac{1}{3}(1+\delta_1) \\ P_n^h = fl(fl(\frac{10}{3})P_{n-1}^h - P_{n-2}^h) \end{cases}$$

Suppose $fl(\frac{10}{3}) = \frac{10}{3}$, and $fl(\frac{10}{3}P_{n-1}^h - P_{n-2}^h) = \frac{10}{3}P_{n-1}^h - P_{n-2}^h$

$\Rightarrow P_n^h = \frac{10}{3}P_{n-1}^h - P_{n-2}^h$ (假設$\frac{10}{3}$可以算的很準，且每次算P_n^h都沒有誤差，則P_n^h的行為就會像P_n^e一樣)

Let $e_n = P_n^h - P_n^e \Rightarrow \begin{cases} e_0 = 0 \\ e_1 = \frac{1}{3}\delta_1 \\ e_n = \frac{10}{3}e_{n-1} - e_{n-2} \end{cases}$

$\Rightarrow e_n = \frac{\delta}{8}(3)^n$ (unstable) ※

5.

$f(x \pm nh) = f(x) \pm f'(x) \cdot nh + f''(x) \cdot \frac{(nh)^2}{2!} \pm f^{(3)}(x) \cdot \frac{(nh)^3}{3!} + f^{(4)}(x)\frac{(nh)^4}{4!} \pm \cdots$

若要消除 $f(x)$, $f''(x)$, $f^{(3)}(x)$, $f^{(4)}(x)$，且保留 $f'(x)$

則需要5個方程 $f(x)$, $f(x \pm h)$, $f(x \pm 2h)$

① $f(x+h) - f(x-h) = 2(f'(x)h + f^{(3)}(x)\frac{h^3}{6} + f^{(5)}(x)\frac{h^5}{5!} + \cdots\cdots)$

② $f(x+2h) - f(x-2h) = 2(f'(x) \cdot 2h + f^{(3)}(x)\frac{8h^3}{6} + f^{(5)}(x)\frac{(2h)^5}{5!} + \cdots\cdots)$

①×6 − ②×$\frac{6}{8}$ $= (12-3)f'(x) \cdot h + \underbrace{2(1-1)}_{=0}f^{(3)}(x) \cdot h^3 + O(h^5)$

$\Rightarrow \frac{①\times 6 - ②\times \frac{6}{8}}{9h} = f'(x) + O(h^4)$ Need 4 points

$\Rightarrow \frac{\frac{1}{3}(f(x+h) - f(x-h)) - \frac{1}{12}(f(x+2h) - f(x-2h))}{h} = f'(x) + O(h^4)$ ※