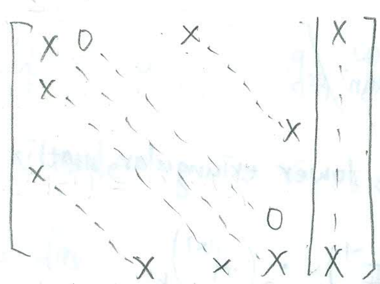
1. (a) (i) Input : A , b & Gauss elimination for agumented matrix $[A|b]$

```
for i = 1, ...... n-1
    for j = i+1, ......, n
        m_ij = a_ji / a_ii ;
        a_ij = 0 ;
        for k = i+1, ....., n
            a_jk = a_jk - m_ij a_ik ;
        end
        b_j = b_j - m_ij b_i ;
    end
end
```

(ii) $N^3/3$

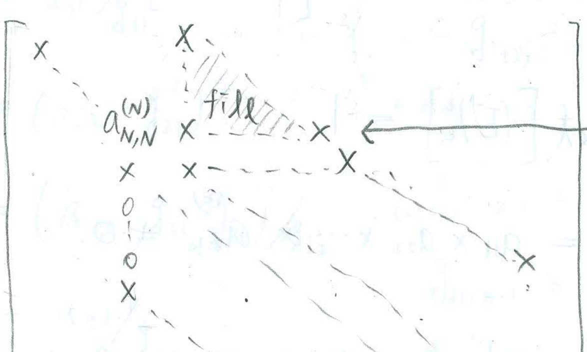
(b) $A=$



do Gaussian elimination

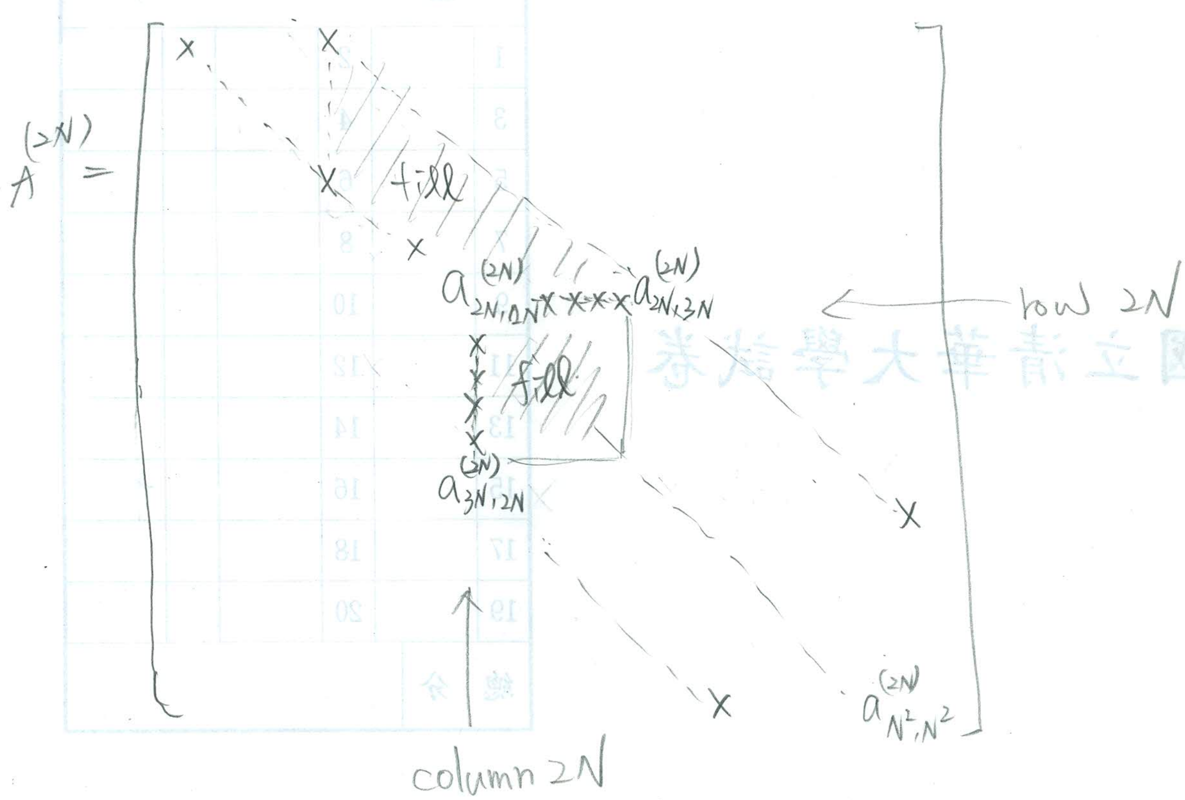
After $N-1$ steps.

$A^{(N)}=$



$a^{(N)}_{N,N}$, $a^{(N)}_{N,N+1}$, ... , $a^{(N)}_{N,2N}$ are nonzero , row N has $N+1$ nonzero entries

After $2N-1$ steps



Both row $2N$ and column $2N$ has $N+1$ nonzero entries

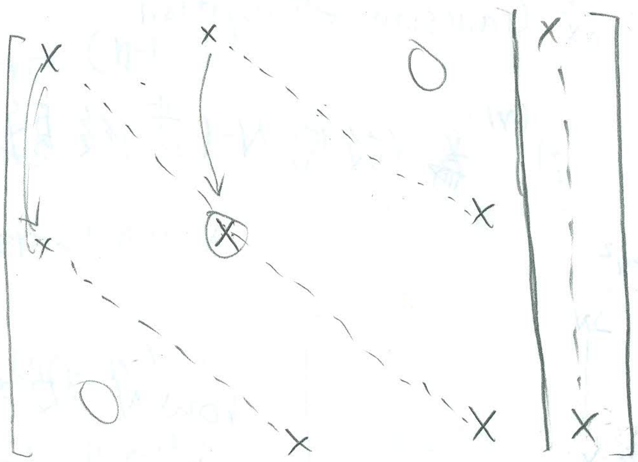
so, In step $2N$, we need N^2 multip.

and from "step $2N$" to "step N^2-N"

we need $N^2(N^2-N-2N+1) = N^4 + \cdots$

Hence the leading order is N^4 ※

1.(c)

$[A|b] =$  $_{N^2\times(N^2+1)}$

在第一步時消去 $a^{(1)}_{N+1,1}$，

$\Rightarrow a^{(2)}_{N+1,1} = 0,\quad a^{(2)}_{N+1,N+1} = a^{(1)}_{N+1,N+1} - \dfrac{a^{(1)}_{N+1,1}}{a^{(1)}_{11}}\, a^{(1)}_{1,N+1}$

$\Rightarrow$ 與(b)小題不一樣，不會增加某一列的entry個數，
也不會增加某一行的entry個數

So, 每一步需要 1+1+1 個乘除

$\Rightarrow$ leading order is $\underline{3N^2}$ ∦

2. $\because \det A_2 = \det\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix} = 0$

& $B(1,1) = 0$

So, only matrix C admit LU decomposition

and $C = \begin{bmatrix}2 & -1 & 1\\ 3 & 3 & 9\\ 3 & 3 & 5\end{bmatrix} \begin{matrix} \\ -3/2 \\ -3/2\end{matrix} \longrightarrow \begin{bmatrix}2 & -1 & -1\\ 0 & 9/2 & 15/2\\ 0 & 9/2 & 7/2\end{bmatrix} \begin{matrix} \\ \\ -1\end{matrix} \longrightarrow \begin{bmatrix}2 & -1 & 1\\ 0 & 9/2 & 15/2\\ 0 & 0 & -4\end{bmatrix} = U$

$\Rightarrow C = \begin{bmatrix}1 & 0 & 0\\ 3/2 & 1 & 0\\ 3/2 & 1 & 1\end{bmatrix} U$ ∦

3. True,

proof by mathmatical induction:

① $n=1$, $A_1=[a_{11}]$, $\det A_1 = a_{11} \neq 0$

② Assuming each pivot element $a_{ii}^{(i)} \neq 0$ for $i=1,2,\dots,k-1$

$$A^{(k)} = \begin{pmatrix} a_{11}^{(1)} & & & & & \\ & \ddots & & & & \\ 0 & & a_{k-1,k-1}^{(k-1)} & & & \\ 0 & \cdots & 0 & a_{k,k}^{(k)} & & \\ \vdots & & \vdots & x & & \\ \vdots & & \vdots & \vdots & & \\ 0 & \cdots & 0 & x & & a_{nn}^{(k)} \end{pmatrix}$$

, $EA = A^{(k)}$, where E is some lower triangular matrix, $E(i,i) = I$.

$\Rightarrow A = E^{-1}A^{(k)} \Rightarrow A_k = (E^{-1})_k \cdot (A^{(k)})_k$

$\because \det A_k = \det\left[(E^{-1})_k\right] \cdot \det\left[(A^{(k)})_k\right] \neq 0$

& $\det\left[(E^{-1})_k\right] = 1$

$\Rightarrow \det\left[(A^{(k)})_k\right] = a_{11} \times a_{22}^{(2)} \times \cdots \times a_{kk}^{(k)} \neq 0$

$\Rightarrow a_{kk}^{(k)} \neq 0$, by induction $\Rightarrow$ G-E on A does not require pivoting

$\because$ Gaussian elimination on A does not require pivoting

$\therefore$ A admits an LU decomposition ※

4. (1) look chap06 slides (new) p75~p76 ※

(2) $A = GG^t$, Try compute:

$$\begin{pmatrix} g_{11} & & & & \\ g_{21} & g_{22} & & 0 & \\ g_{31} & g_{32} & g_{33} & & \\ 0 & g_{42} & g_{43} & g_{44} & \\ 0 & 0 & g_{53} & g_{54} & g_{55} \end{pmatrix} \begin{pmatrix} g_{11} & g_{21} & g_{31} & 0 & 0 \\ & g_{22} & g_{32} & g_{42} & 0 \\ & & g_{33} & g_{43} & g_{53} \\ & & & g_{44} & g_{54} \\ & & & & g_{55} \end{pmatrix}$$

$$= \begin{pmatrix} g_{11}^2 & g_{11}\cdot g_{21} & g_{11}\cdot g_{31} & 0 & 0 \\ g_{11}\cdot g_{21} & g_{21}^2+g_{22}^2 & g_{21}\cdot g_{31}+g_{32}\cdot g_{22} & g_{22}\cdot g_{42} & 0 \\ g_{11}\cdot g_{31} & g_{31}\cdot g_{21}+g_{32}\cdot g_{22} & g_{31}^2+g_{32}^2+g_{33}^2 & g_{32}\cdot g_{42}+g_{33}\cdot g_{43} & g_{43}\cdot g_{53} \\ 0 & g_{22}\cdot g_{42} & g_{32}\cdot g_{42}+g_{33}\cdot g_{43} & g_{42}^2+g_{43}^2+g_{44}^2 & g_{43}\cdot g_{53}+g_{44}\cdot g_{54} \\ 0 & 0 & g_{33}\cdot g_{53} & g_{43}\cdot g_{53}+g_{44}\cdot g_{54} & g_{53}^2+g_{54}^2+g_{55}^2 \end{pmatrix}$$

pseudo-code:

$g_{11} = \sqrt{a_{11}}$;

$g_{21} = a_{21}/g_{11}$;

$g_{31} = a_{31}/g_{11}$;

$g_{22} = (a_{22} - g_{21}^2)^{1/2}$

$g_{32} = (a_{32} - g_{31}\cdot g_{21})/g_{22}$

$g_{42} = a_{42}/g_{22}$

for $i = 3 : n-2$

$g_{i,i} = (a_{i,i} - g_{i,i-1}^2 - g_{i,i-2}^2)^{1/2}$;

$g_{i+1,i} = (a_{i+1,i} - g_{i,i-1}\cdot g_{i+1,i-1})/g_{i,i}$;

$g_{i+2,i} = a_{i+2,i}/g_{i,i}$;

end

$g_{n-1,n-1} = (a_{n-1,n-1} - g_{n-1,n-2}^2 - g_{n-1,n-3}^2)^{1/2}$;

$g_{n,n-1} = (a_{n,n-1} - g_{n-1,n-2}\cdot g_{n,n-2})/g_{n-1,n-1}$

$g_{n,n} = (a_{nn} - g_{n,n-1}^2 - g_{n,n-2}^2)^{1/2}$; ※

M/D:

"$\sqrt{\ }$" : N

"×" : $6 + 5(N-2-3+1) + 6$

$\Rightarrow$ need $10N + 5N - 8 = 15N - 8$ multip. ※

5. (a) look textbook. 7.3. P437 ~ 449.

(b) Let $\lambda_1, \lambda_2, \dots, \lambda_r$ are eigenvalues of T_J, $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_r|$

$\because$ T_J is diagonalizable.

$\therefore$ for any $x_0 \in \mathbb{R}^n$, $x_0 = v_1 + v_2 + \cdots + v_r$, for some $v_i \in V(\lambda_i)$, $V(\lambda_i)$ is eigenspac

then $\|T^k x_0\|^{\frac{1}{k}} = \| \lambda_1^k v_1 + \lambda_2^k v_2 + \cdots + \lambda_r^k v_r \|^{\frac{1}{k}}$

$= |\lambda_1| \left\| \left(\frac{\lambda_1}{|\lambda_1|}\right)^k v_1 + \left(\frac{\lambda_2}{|\lambda_1|}\right)^k v_2 + \cdots + \left(\frac{\lambda_r}{|\lambda_1|}\right)^k v_r \right\|^{\frac{1}{k}} \approx |\lambda_1| \, \|v_1\|^{\frac{1}{k}} \approx |\lambda_1| = \rho(A)$, when k sufficiently large

So, $\|x^{(k)} - x\| = \| T^k (x^{(0)} - x)\| \le \|T^k\| \, \|e^{(0)}\| = \|T^k\| \approx \rho(A)^k$, where $x = T_J x + c_J$

$\Rightarrow \|e^{(k)}\| \le \rho(A)^k \overset{\text{hope}}{\le} 10^{-4}$

$\Rightarrow k \geq -4 / \log_{10} 0.94 \approx 148.8530 \Rightarrow k = 149$ #

(c) Use SOR-Jacobi, with $\omega = -2/3$

$x^{(k)} = T_{\omega, J} x^{(k-1)} + \omega D^{-1} b$

$= \left[(I - \omega) I + \omega T_J \right] x^{(k-1)} + c_J$

we hope $-1 < \lambda(T_{\omega,J}) < 1$

and find $\omega \ni \rho(T_{\omega,J}) = \min_{\omega'} \{ \rho(T_{\omega',J}) \}$

$\because T_{\omega,J} = (1+\omega) I + \omega T_J$

i.e. A linear transformation $L: \lambda(T_J) \to \underbrace{\omega \lambda(T_J) + (1-\omega)}_{= \lambda(T_{\omega,J})}$

The center of domain [2,3] is 2.5

if $L(2.5) = 0 \Rightarrow \omega = -2/3$

$\Rightarrow$ when $\omega = -2/3$, $-1/3 \le \lambda(T_{\omega,J}) \le 1/3$

& $\frac{1}{3} = \min_{\omega'} \{ \rho(T_{\omega',J}) \}$

#

6, (i) unique:

Suppose both

$B = U_1 L_1$ and $B = U_2 L_2$ are UL factorizations.

Since B is nonsingular, U_1, L_1, U_2, L_2 are all nonsingular.

and. $B = U_1 L_1 = U_2 L_2 \Rightarrow U_2^{-1} U_1 = L_2 L_1^{-1}$.

Since U_1 and U_2 are unit upper triangular,

$\Rightarrow U_2^{-1} U_1$ is also unit upper triangular.

On the other hand, since L_1 and L_2 are lower triangular, $L_2 L_1^{-1}$ is also lower triangular.

Therefore, $U_2^{-1} U_1 = I = L_2 L_1^{-1}$

$\Rightarrow U_1 = U_2$ and $L_1 = L_2$ #

(ii) psoudo-code: (B: tridiagonal, $\tilde{U}\tilde{L}$, $u_{ii} = 1$)

相當於由下往上做 Gaussian elimination

try to compute
$$\tilde{U}\tilde{L} = \begin{pmatrix} 1 & u_{12} & 0 & 0 \\ 0 & 1 & u_{23} & 0 \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & 0 & l_{43} & l_{44} \end{pmatrix}$$

$$= \begin{pmatrix} l_{11} + u_{12} \cdot l_{21} & u_{12} \cdot l_{22} & 0 & 0 \\ l_{21} & l_{22} + u_{23} \cdot l_{32} & u_{23} \cdot l_{33} & 0 \\ 0 & l_{32} & l_{33} + u_{34} \cdot l_{34} & u_{34} \cdot l_{44} \\ 0 & 0 & l_{43} & l_{44} \end{pmatrix}$$

$L_1 = zeros(n, 1);$

$L_2 = zeros(n-1, 1);$

$U_1 = ones(n, 1);$

$U_2 = zeros(n-1, 1);$

$L_1(n) = B(n, n);$

$L_2(n-1) = B(n, n-1);$

$U_2(n-1) = B(n-1, n) / L_1(n);$

```
for i = n-1 : -1 : 2
    L1(i) = B(i,i) - L2(i) . U2(i);
    L2(i-1) = B(i, i-1);
    U2(i-1) = B(i-1, i) / L1(i);
end
```

$L_1(1) = B(1,1) - L_2(1) \cdot U_2(1);$

$\tilde{U} = diag(U_1) + diag(U_2, 1);$

$\tilde{L} = diag(L_1) + diag(L_2, -1);$

$\% A = \tilde{U}\tilde{L}$

$\sum_{i=1}^{10} abs(\tilde{\ell}_{ii}) = sum(abs(L_1)) = 59.4874$ #

if $A = LU$, $\sum_{i=1}^{10} abs(u_{ii}) = 59.4874$