1. (a) $-3.5 = -1.11 \times 2^1 = \underset{s}{1}\ \underset{c}{\underline{1000}}\ \underset{f}{\underbrace{1100000}}$

$$\text{"}(-1)^s 2^{c-x}(1+f)\text{"}, \quad \because 8 - x = 1 \Rightarrow x = 7$$

$$0.625 = 1.01 \times 2^{-1} = (-1)^0 2^{6-7}(1+0.01)$$

$$= 0\ \underline{0110}\ 0100000 \quad \#$$

(b) Let $x = 1.a_1 a_2 \cdots a_7 a_8 \cdots \times 2^e$

$$\left|\frac{x - fl(x)}{x}\right| = \frac{0.\overbrace{00\cdots0}^{7} a_8 a_9 \cdots \times \cancel{2^e}}{1.a_1 a_2 \cdots a_7 a_8 a_9 \cdots \times \cancel{2^e}}$$

$$\leq \frac{0.\overbrace{00\cdots0}^{6}100\cdots}{1.0\cdots\cdots} = 2^{-7} \quad \#$$

2. $(a-b)(a^2+ab+b^2) = a^3 - \cancel{a^2b} + \cancel{a^2b} - \cancel{ab^2} + \cancel{ab^2} - b^3$

$$= a^3 - b^3$$

$$\Rightarrow (a-b) = \frac{a^3 - b^3}{a^2+ab+b^2}$$

$$\Rightarrow (100002)^{\frac{1}{3}} - (100001)^{\frac{1}{3}} = \frac{1}{(100002)^{\frac{2}{3}} + (100002)^{\frac{1}{3}}(100001)^{\frac{1}{3}} + (100001)^{\frac{2}{3}}}$$

$$= 1.547180806108708 \times 10^{-4} \quad \#$$

use vpa : $0.000\underline{15471808061087078}627355712391 53$

第16位!

3. $\begin{cases} P_0^e = 1/3,\ P_1^e = 1/6,\ P_2^e = \frac{1}{12} \\ P_n^e = \frac{7}{2} P_{n-1}^e - \frac{7}{2} P_{n-2}^e + P_{n-3}^e \end{cases}$

solve : $x^3 - \frac{7}{2}x^2 + \frac{7}{2}x - 1 = 0$

$\Rightarrow x = 1/2,\ 1,\ 2$

Assume $P_n^e = c_1\left(\frac{1}{2}\right)^n + c_2(1)^n + c_3(2)^n$

$$\Rightarrow \begin{cases} P_0^e = \frac{1}{3} = c_1 + c_2 + c_3 & \text{--- ①} \\ P_1^e = \frac{1}{6} = \frac{1}{2}c_1 + c_2 + 2c_3 & \text{--- ②} \\ P_2^e = \frac{1}{12} = \frac{1}{4}c_1 + c_2 + 4c_3 & \text{--- ③} \end{cases}$$

$\Rightarrow c_1 = \frac{1}{3},\ c_2 = c_3 = 0$

$\Rightarrow$ The exact solution $P_n^e = \frac{1}{3}\left(\frac{1}{2}\right)^n$ ※

Numerical solution :

$$\begin{cases} P_0^h = fl\left(\frac{1}{3}\right) = \frac{1}{3}(1+\delta_0) \\ P_1^h = fl(1/6) = \frac{1}{6}(1+\delta_1) \\ P_2^h = fl\left(\frac{1}{12}\right) = \frac{1}{12}(1+\delta_2) \\ P_n^h = fl\left(fl\left(\frac{7}{2}\right)P_{n-1}^h - fl\left(\frac{7}{2}\right)P_{n-2}^h + P_{n-3}^h\right) \end{cases}$$

$$= (1+\delta_n)\left(\frac{7}{2}P_{n-1}^h - \frac{7}{2}P_{n-2}^h + P_{n-3}^h\right)$$

Rewrite $P_n^e = (1+\delta_n)\left(\frac{7}{2}P_{n-1}^e - \frac{7}{2}P_{n-2}^e + P_{n-3}^e\right) - \delta_n(P_n^e)$

Equation of error :

$$e_n = P_n^h - P_n^e = (1+\delta_n)\left(\frac{7}{2}e_{n-1} - \frac{7}{2}e_{n-2} + e_{n-3}\right) + \delta_n(P_n^e)$$

$$\Rightarrow e_n = \tilde{e}_n + \tilde{\tilde{e}}_n$$

where
$$\begin{cases} \tilde{e}_0 = \frac{1}{3}\delta_0, \ \tilde{e}_1 = \frac{1}{6}\delta_1, \ \tilde{e}_2 = \frac{1}{12}\delta_2 \\ \tilde{e}_n = (1+\delta_n)\left(\frac{7}{2}\tilde{e}_{n-1} - \frac{7}{2}\tilde{e}_{n-2} + \tilde{e}_{n-3}\right) \end{cases}$$

$$\begin{cases} \tilde{\tilde{e}}_0 = 0, \ \tilde{\tilde{e}}_1 = 0, \ \tilde{\tilde{e}}_2 = 0 \\ \tilde{\tilde{e}}_n = (1+\delta_n)\left(\frac{7}{2}\tilde{\tilde{e}}_{n-1} - \frac{7}{2}\tilde{\tilde{e}}_{n-2} + \tilde{\tilde{e}}_{n-3}\right) + \frac{\delta_n}{3}\left(\frac{1}{2}\right)^n \end{cases}$$

we try to compute $\tilde{e}_3 = (1+\delta_3)\left(\frac{7}{2}\cdot\frac{1}{12}\delta_2 - \frac{7}{2}\cdot\frac{1}{6}\delta_1 + \frac{1}{3}\delta_0\right)$

Because $|\delta_i| \leq \delta_M < 10^{-15}$ for each i

$\Rightarrow \delta_i\delta_j < 10^{-30}$ (we neglect it) for each i, j

Hence, the solution $\tilde{e}_n$ can be approximated by assuming $\delta_n = 0$

$$\Rightarrow \tilde{e}_n \approx \frac{7}{2}\tilde{e}_{n-1} - \frac{7}{2}\tilde{e}_{n-2} + \tilde{e}_{n-3}$$

Assume $\tilde{e}_n = d_1\left(\frac{1}{2}\right)^n + d_2(1)^n + d_3(2)^n$

$$\Rightarrow \begin{cases} \frac{1}{3}\delta_0 = d_1 + d_2 + d_3 \\ \frac{1}{6}\delta_1 = \frac{1}{2}d_1 + d_2 + 2d_3 \\ \frac{1}{12}\delta_2 = \frac{1}{4}d_1 + d_2 + 4d_3 \end{cases}$$

$$\Rightarrow d_3 = -\frac{16}{21}\left(\frac{1}{12}\delta_2 - \frac{1}{12}\delta_0 - \frac{3}{12}\delta_1 + \frac{3}{12}\delta_0\right) \neq 0 \text{ in general}$$

$$\Rightarrow \tilde{e}_n \approx d_3(2)^n$$

$\tilde{\tilde{e}}_n$ is more complicated, and grows at same rate with $\tilde{e}_n$

relative error: $\dfrac{|e_n|}{p_n^e} \approx C\dfrac{2^n}{2^{-n}} = C4^n$ unstable #

4. ① $f(x-h) = f(x) - f'(x)h + f''(x)\frac{h^2}{2} - f^{(3)}(\xi_1)\frac{h^3}{6}$

② $f(x) = f(x)$

③ $f(x+2h) = f(x) + f'(x)\cdot 2h + f''(x)\cdot 2h^2 + f^{(3)}(\xi_2)\cdot\frac{4}{3}h^3$

where $\xi_1 \in (x-h, x)$, $\xi_2 \in (x, x+2h)$

(a) (i) approx. $f'(x)$:

③ $- 4\times$ ① $+ 3\times$ ②

$$= (1-4+3) f(x) + (2h+4h+0) f'(x) + (2h^2-2h^2+0) f''(x)$$

$$+ \frac{4}{3}h^3 f^{(3)}(\xi_2) + \frac{2}{3}h^3 f^{(3)}(\xi_1)$$

$$= 6h f'(x) + 2 f^{(3)}(\xi) h^3$$

by intermediate value theorem $\exists\, \xi \in (x-h, x+2h)$

$$\Rightarrow \frac{f(x+2h) + 3f(x) - 4f(x-h)}{6h} = f'(x) + \underbrace{\frac{1}{3} f^{(3)}(\xi) h^2}_{\text{error identity}}$$

(ii) approx. $f''(x)$:

③ $+ 2\times$ ① $- 3\times$ ②

$$= (1+2-3) f(x) + (2h-2h) f'(x) + (2h^2+h^2) f''(x)$$

$$+ \frac{f^{(3)}(\xi_2)4h^3}{3} - \frac{f^{(3)}(\xi_1)h^2}{3}$$

$$\Rightarrow \frac{f(x+2h) - 3f(x) + 2f(x-h)}{3h^2} = f''(x) + \frac{4}{9} f^{(3)}(\xi_2)h - \frac{1}{9} f^{(3)}(\xi_1)h$$

(b) error identity : $\frac{1}{3} f^{(3)}(\xi) h^2$, $\xi \in (x-h, x+2h)$

(c) Let $E(h) = \dfrac{f(x+2h) - 3f(x) + 2f(x-h)}{3} - h^2 f''(x)$, $E(0) = 0$

$$E'(h) = \frac{2f'(x+2h) - 2f'(x-h)}{3} - 2h f''(x) \quad , \quad E'(0) = 0$$

$$E''(h) = \frac{4f''(x+2h) + 2f''(x-h)}{3} - 2f''(x) \quad , \quad E''(0) = 0$$

$$E^{(3)}(h) = \frac{8f^{(3)}(x+2h) - 2f^{(3)}(x-h)}{3} \quad , \quad E^{(3)}(0) \neq 0$$

$$E^{(4)}(h) = \frac{16f^{(4)}(x+2h) + 2f^{(4)}(x-h)}{3}$$

$$E(h) = \underset{0}{\underset{\parallel}{E(0)}} + \underset{0}{\underset{\parallel}{E'(0)h}} + \underset{0}{\underset{\parallel}{E''(0)}}\frac{h^2}{2} + E^{(3)}(0)\frac{h^3}{3!} + \int_0^h \frac{E^{(4)}(t)(h-t)^3}{3!} dt$$

$$= E^{(3)}(0) \cdot \frac{h^3}{3!} - E^{(4)}(\xi') \frac{(h-t)^4}{4!}\Big|_0^h$$

$$= E^{(3)}(0) \frac{h^3}{3!} + E^{(4)}(\xi') \frac{h^4}{4!} = \frac{2f^{(3)}(x)h^3}{3!} + \frac{16f^{(4)}(x+2\xi') + 2f^{(4)}(x-\xi')}{3} \frac{h^4}{4!}$$

$$= \frac{1}{3} f^{(3)}(x) h^3 + \frac{1}{4} f^{(4)}(\xi) h^4 \text{, where } \xi \in (x-\xi', x+2\xi')$$

$$\Rightarrow \frac{f(x+2h) - 3f(x) + 2f(x-h)}{3h^2} - f''(x) = \frac{1}{3} f^{(3)}(x) h + \frac{1}{4} f^{(4)}(\xi) h^2$$

*

or by a-(i) the error bound is $\left| \frac{4}{9} f^{(3)}(\xi_2) h - \frac{1}{9} f^{(3)}(\xi_1) h \right|$

$$\leq \frac{5}{9} f^{(3)}(\xi) h$$

*

5, $n=3$. (look textbook P221 ~ P222)

Legendre polynomial $P_3(x) = x^3 - \frac{3}{5}x$

The roots of $P_3(x)$ are $\underline{0, \pm\sqrt{\frac{3}{5}}}$ ※

OR, Assume $\int_{-1}^{1} f(x)dx = c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$ (*)

The formula gives exact results when $f(x) = 1, x, x^2, x^3, x^4$, and x^5

if $f(x)=1 \Rightarrow \int_{-1}^{1} f(x) = 2 = c_1 + c_2 + c_3$

if $f(x)=x \Rightarrow \int_{-1}^{1} x = 0 = c_1 x_1 + c_2 x_2 + c_3 x_3$

$\vdots \qquad \vdots \qquad \vdots$

if $f(x)=x^5 \Rightarrow \int_{-1}^{1} x^5 = 0 = c_1 x_1^5 + c_2 x_2^5 + c_3 x_3^5$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1^3 & x_2^3 & x_3^3 \\ x_1^4 & x_2^4 & x_3^4 \\ x_1^5 & x_2^5 & x_3^5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2/3 \\ 0 \\ 2/5 \\ 0 \end{bmatrix}$$

Consider two systems:

$$(1) \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1^4 & x_2^4 & x_3^4 \end{bmatrix}}_{\text{Let} \parallel A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2/3 \\ 2/5 \end{bmatrix} \ \& \ (2) \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ x_1^3 & x_2^3 & x_3^3 \\ x_1^5 & x_2^5 & x_3^5 \end{bmatrix}}_{\text{Let} \parallel B} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In the three points formula (*), we may assume

$c_1 \neq 0$, $c_2 \neq 0$, $c_3 \neq 0$, and $-1 \leq x_1 < x_2 < x_3 \leq 1$

In system (2), $\because c_1 \neq 0, c_2 \neq 0, c_3 \neq 0$

$$\therefore \det B = 0$$

and $\det B = x_1 x_2 x_3 (x_3^2 - x_2^2)(x_2^2 - x_1^2)(x_3^2 - x_1^2) = 0$

In system (1)

$$\det A = (x_3^2 - x_2^2)(x_2^2 - x_1^2)(x_3^2 - x_1^2)$$

(i) if $\det A \neq 0$

$\because \det B = 0 \Rightarrow x_1 = 0$ or $x_2 = 0$ or $x_3 = 0$

case 1. if $x_1 = 0$ $(0 = x_1 < x_2 < x_3)$

The system (2):
$$\begin{bmatrix} 0 & x_2 & x_3 \\ 0 & x_2^3 & x_3^3 \\ 0 & x_2^5 & x_3^5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_2 & x_3 \\ x_2^3 & x_3^3 \end{bmatrix} \begin{bmatrix} c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Because $c_2 \neq 0, c_3 \neq 0 \Rightarrow \det \begin{pmatrix} x_2 & x_3 \\ x_2^3 & x_3^3 \end{pmatrix} = (\underset{+}{x_2}\underset{+}{x_3})(\underset{+}{x_3^2} - \underset{+}{x_2^2}) = 0$

$\Rightarrow x_3^2 - x_2^2 = 0$ ※ $(\because x_2 \neq x_3, 0 < x_2 < x_3)$

case 2. if $x_2 = 0$ $(x_1 < x_2 = 0 < x_3)$

from system (2) $\Rightarrow \begin{bmatrix} x_1 & x_3 \\ x_1^3 & x_3^3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow (x_1 x_3)(x_3^2 - x_1^2) = 0 \Rightarrow x_3^2 = x_1^2$$

Then we obtain $x_1 = -x_3$ and $x_1 < x_2 = 0 < x_3$

and $\because c_1 x_1 + c_2 x_2 + c_3 x_3 = c_1 x_1 + c_3 x_3 = x_3(c_3 - c_1) = 0$

$\Rightarrow c_3 = c_1$

use system (1)

$$\begin{bmatrix} x_1^2 & x_3^2 \\ x_1^4 & x_3^4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_1^2 & x_1^2 \\ x_1^4 & x_1^4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/5 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 2c_1 x_1^2 = \frac{2}{3} & \cdots\cdots ① \\ 2c_1 x_1^4 = \frac{2}{5} & \cdots\cdots ② \end{cases}$$

$$\frac{②}{①} \Rightarrow x_1^2 = \frac{\frac{2}{5}}{\frac{2}{3}} = \frac{3}{5} \Rightarrow x_1 = \pm\sqrt{\frac{3}{5}}$$

Let $x_1 = -\sqrt{\frac{3}{5}}$, $x_2 = 0$, $x_3 = \sqrt{\frac{3}{5}}$

$$\therefore \begin{cases} c_1 + c_2 + c_3 = 2 \\ \frac{3}{5}c_1 + 0 + \frac{3}{5}c_3 = 2/3 \\ \frac{9}{25}c_1 + 0 + \frac{9}{25}c_3 = 2/5 \end{cases} \Rightarrow c_1 = \frac{5}{9},\ c_3 = \frac{5}{9},\ c_2 = \frac{8}{9}$$

$$\Rightarrow \int_{-1}^{1} f(x)\,dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) \quad \#$$

case 3, if $x_3 = 0$ ($x_1 < x_2 < x_3 = 0$)

similar to case I, we derive to $*$

(ii) if $\det A = 0$

$\Rightarrow x_1^2 = x_3^2$ or $x_1^2 = x_2^2$ or $x_2^2 = x_3^2$

$\Rightarrow x_1 = -x_3$ or $x_1 = -x_2$ or $x_2 = -x_3$

$\Rightarrow \underset{(1)}{x_1 < x_2 = 0 < x_3}$ or $\underset{(2)}{x_1 < 0 < x_2 < x_3}$ or $\underset{(3)}{x_1 < x_2 < 0 < x_3}$

The case(1) is the same as (i)-case I

In case (2) $(x_1 = -x_2 < 0 < x_2 < x_3)$

use system (2)

$$\Rightarrow \begin{bmatrix} -x_2 & x_2 & x_3 \\ -x_2^3 & x_2^3 & x_3^3 \\ -x_2^5 & x_2^5 & x_3^5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ x_2^2 & x_2^2 & x_3^2 \\ x_2^4 & x_2^4 & x_3^4 \end{bmatrix}}_{\text{Let } D} \begin{bmatrix} -x_2 c_1 \\ x_2 c_2 \\ x_3 c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \operatorname{rank}(D) = 2$$

$\exists\ E_1, E_2$ is invertible $\ni$ $E_1 D E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \Lambda$

$\Rightarrow D = E_1^{-1} \Lambda E_2^{-1}$,

Let $W = E_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow DW = E_1^{-1} \Lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$

$\Rightarrow W \in \ker(D)$

$\Rightarrow \exists\ v \neq 0\ \&\ k \neq 0,\ Dv = 0 \ni W = kv$

$$\Rightarrow E_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = kv \Rightarrow \begin{pmatrix} 0 \\ 0 \\ E_2(3,3) \end{pmatrix} = kv \Rightarrow v = \begin{pmatrix} 0 \\ 0 \\ \frac{E_2(3,3)}{k} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -x_2 c_1 \\ x_2 c_2 \\ x_3 c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{E_2(3,3)}{k} \end{pmatrix} \xrightarrow{\because x_2 \neq 0} c_1 = c_2 = 0 \quad \rightarrow\!\!\!\!\times$$

Note that: $\because E_2$ is invertible $\Rightarrow E_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \neq 0$

The case (3) is the same as case (2) ##

6. (a) look HW4, 4~5

$$\log_2\left(\frac{I-I_{2h}}{I-I_h}\right) = P_1 \approx 1.5 \quad \#$$

(b)
$$M = N_1(h) + C_1 h^{1.5} + C_2 h^{P_2} + \cdots\cdots$$

$$M = N_1\left(\tfrac{h}{2}\right) + C_1\left(\tfrac{h}{2}\right)^{1.5} + C_2\left(\tfrac{h}{2}\right)^{P_2} + \cdots\cdots$$

$$N_2(h) = \frac{2^{1.5} N_1\left(\frac{h}{2}\right) - N_1(h)}{2^{1.5}-1}$$

Let $h = \frac{1}{N}$

$$N(h) = N\left(\tfrac{1}{N}\right) = \sum_{j=0}^{N-1} \frac{1}{N} f(x_{j+\frac{1}{2}}), \text{ where } x_j = \frac{j}{N} \text{ and } x_{j+\frac{1}{2}} = \frac{j}{N} + \frac{1}{2N}$$

$$N\left(\tfrac{h}{2}\right) = N\left(\tfrac{1}{2N}\right) = \sum_{i=0}^{2N-1} \frac{1}{2N} f(x'_{i+\frac{1}{2}}), \text{ where } x'_i = \frac{i}{2N} \text{ and } x'_{i+\frac{1}{2}} = \frac{i}{2N} + \frac{1}{4N}$$

$$\Rightarrow N_2(h) = N_2\left(\tfrac{1}{N}\right) = \frac{2^{1.5} \cdot \sum_{i=0}^{2N-1} \frac{1}{2N} f(x'_{i+\frac{1}{2}}) - \sum_{j=0}^{N-1} \frac{1}{N} f(x_{j+\frac{1}{2}})}{2^{1.5}-1}$$

$$= \frac{h}{2^{1.5}-1}\left(\sqrt{2}f\left(\tfrac{h}{4}\right) - f\left(\tfrac{h}{2}\right) + \sqrt{2}f\left(\tfrac{3h}{4}\right) + \sqrt{2}f\left(\tfrac{5h}{4}\right) - f\left(\tfrac{3h}{2}\right) + \sqrt{2}f\left(\tfrac{7}{4}h\right) + \cdots\cdots\right)$$

$$\overset{\text{Let } c = 2^{1.5}-1}{=\!=} \sum_{j=0}^{N-1} h\left(\frac{\sqrt{2}}{c} f(jh + \tfrac{1}{4}h) - \frac{1}{c} f(jh + \tfrac{1}{2}h) + \frac{\sqrt{2}}{c} f(jh + \tfrac{3}{4}h)\right) \quad \#$$

6 (c) midpoint rule : $\int_a^b f(x)\,dx = (b-a)\, f\left(\frac{a+b}{2}\right) + \frac{h^3}{24} f''(\xi)$

where $\xi \in (a, b)$

Composite midpoint rule :

$$I = \int_0^1 f(x)dx = \sum_{i=0}^{N-1} h\, f(x_{i+\frac{1}{2}}) + \sum_{i=0}^{N-1} \frac{h^3}{24} f''(\xi_i)$$

Note :

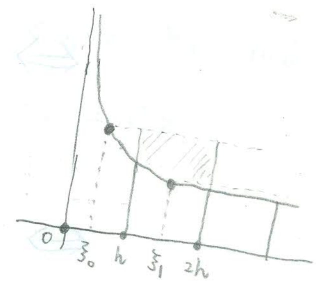
$f(x) = x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$

$f''(x) = \frac{-1}{4} x^{-\frac{3}{2}}$

where $h = \frac{1}{N}$, $x_i = \frac{i}{N}$, $x_{i+\frac{1}{2}} = \frac{i}{N} + \frac{1}{2N}$, $\xi_i \in (x_i, x_{i+1})$

$$|I - N_1(h)| = \left| \sum_{i=0}^{N-1} \frac{h^3}{24} f''(\xi_i) \right|$$



$$= \frac{h^3}{24} \sum_{i=0}^{N-1} \frac{1}{4} (\xi_i)^{-\frac{3}{2}}$$

$$\geq \frac{h^3}{24} \int_h^1 \frac{1}{4} x^{-\frac{3}{2}} dx = \frac{h^3}{24} \left(-\frac{1}{2} x^{-\frac{1}{2}} \Big|_h^1 \right)$$

$$= \frac{1}{48} h^{1.5} - \frac{1}{48} h^2$$

$$|I - N_1(h)| = \left| \int_0^h \sqrt{x}\, dx - h f(x_{\frac{1}{2}}) + \int_h^1 \sqrt{x}\, dx - \sum_{i=1}^{N-1} h f(x_{i+\frac{1}{2}}) \right|$$

$$= \left| \frac{2}{3} x^{\frac{3}{2}} \Big|_0^h - h \cdot \left(\frac{h}{2}\right)^{\frac{1}{2}} + \sum_{i=1}^{N-1} \frac{h^3}{24} f''(\xi_i) \right|$$

$$\leq \left| \frac{2}{3} - \frac{1}{\sqrt{2}} \right| h^{1.5} + \left| \sum_{i=1}^{N-1} \frac{h^3}{24} f''(\xi_i) \right| \leq \left| \frac{2}{3} - \frac{1}{\sqrt{2}} \right| h^{1.5} + \frac{h^2}{24} \sum_{i=1}^{N-1} (ih)^{-\frac{3}{2}} \cdot \frac{h}{4}$$

by (*) $$\leq \left| \frac{2}{3} - \frac{1}{\sqrt{2}} \right| h^{1.5} + \frac{h^2}{24} \int_{h/2}^1 \frac{1}{4} x^{-\frac{3}{2}} dx$$

$$= \left| \frac{2}{3} - \frac{1}{\sqrt{2}} \right| h^{1.5} + \frac{h^2}{24} \left(\frac{-1}{2} x^{-\frac{1}{2}} \Big|_{h/2}^1 \right)$$

$$= \left| \frac{2}{3} - \frac{1}{\sqrt{2}} \right| h^{1.5} + \frac{\sqrt{2}}{48} h^{1.5} - \frac{1}{48} h^2 = \left(\left| \frac{2}{3} - \frac{1}{\sqrt{2}} \right| + \frac{\sqrt{2}}{48} \right) h^{1.5} - \frac{1}{48} h^2$$

(**) Claim : $h(mh)^{-\frac{3}{2}} \le \int_{(m-\frac{1}{2})h}^{(m+\frac{1}{2})h} x^{-\frac{3}{2}}\, dx$

$$\Leftrightarrow \quad m^{-\frac{3}{2}} h^{-\frac{1}{2}} \le -2x^{-\frac{1}{2}} \Big|_{(m-\frac{1}{2})h}^{(m+\frac{1}{2})h}$$

$$\Leftrightarrow \quad m^{-\frac{3}{2}} \cancel{h^{-\frac{1}{2}}} \le 2\left(\left(m-\tfrac{1}{2}\right)^{-\frac{1}{2}} - \left(m+\tfrac{1}{2}\right)^{-\frac{1}{2}} \right) \cancel{h^{-\frac{1}{2}}}$$

$$\Leftrightarrow \quad \frac{m^{-\frac{3}{2}}}{2} \le \frac{1}{\sqrt{m-\frac{1}{2}}} - \frac{1}{\sqrt{m+\frac{1}{2}}}$$

$$\Leftrightarrow \quad \frac{m^{-\frac{3}{2}}}{2} \le \frac{\sqrt{m+\frac{1}{2}} - \sqrt{m-\frac{1}{2}}}{\sqrt{m^2-\frac{1}{4}}}$$

$$\Leftrightarrow \quad \frac{m^{-\frac{3}{2}}}{2}\left(\sqrt{m+\tfrac{1}{2}} + \sqrt{m-\tfrac{1}{2}} \right) \le \frac{\cancel{m}+\frac{1}{2} - \cancel{m}+\frac{1}{2}}{\sqrt{m^2-\frac{1}{4}}} = \frac{1}{\sqrt{m^2-\frac{1}{4}}}$$

$$\Leftrightarrow \quad \frac{m^{-3}}{4}\left(m+\cancel{\tfrac{1}{2}}+m-\cancel{\tfrac{1}{2}}+2\cdot\sqrt{m^2-\tfrac{1}{4}} \right) \le \frac{1}{m^2-\frac{1}{4}}$$

$$\Leftrightarrow \quad \frac{1}{m^2}\left(\frac{1}{2} + \frac{\sqrt{1-\frac{1}{4m^2}}}{2} \right) \le \frac{1}{m^2}\left(\frac{1}{2}+\frac{1}{2}\right) = \frac{1}{m^2} \le \frac{1}{m^2-\frac{1}{4}}$$

for all $m \ge 1$. #

by (**) $\Rightarrow$ $\underbrace{\sum_{m=1}^{N-1} h(mh)^{-\frac{3}{2}} \le \int_{\frac{1}{2}h}^{(N-\frac{1}{2})h} x^{-\frac{3}{2}}\, dx \le \int_{\frac{1}{2}h}^{1} x^{-\frac{3}{2}}\, dx}_{(*)}$

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