Numerical Solutions of Nonlinear Systems of Equations

Exercise Set 10.1, page 605

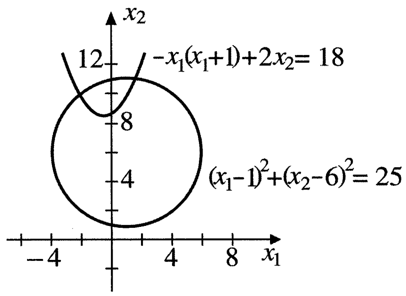
1. Use Theorem 10.5

2. One example is $\mathbf{F}(x_1, x_2) = \left(1, \frac{1}{|x_1-1|+|x_2|}\right)^t$.

3. Use Theorem 10.5 for each of the partial derivatives.

4. The solutions are near $(-1.5, 10.5)$ and $(2, 11)$.

   (a) The graphs are shown in the figure below.



   (b) Use

$$\mathbf{G}_1(\mathbf{x}) = \left(-0.5 + \sqrt{2x_2 - 17.75}, 6 + \sqrt{25 - (x_1 - 1)^2}\right)^t$$

and

$$\mathbf{G}_2(\mathbf{x}) = \left(-0.5 - \sqrt{2x_2 - 17.75}, 6 + \sqrt{25 - (x_1 - 1)^2}\right)^t.$$

For $\mathbf{G}_1(\mathbf{x})$ with $\mathbf{x}^{(0)} = (2, 11)^t$, we have $\mathbf{x}^{(9)} = (1.5469466, 10.969994)^t$, and for $\mathbf{G}_2(\mathbf{x})$ with $\mathbf{x}^{(0)} = (-1.5, 10.5)$, we have $\mathbf{x}^{(34)} = (-2.000003, 9.999996)^t$.

5. (a) Continuity properties can be easily shown. Moreover,

$$\frac{8}{10} \leq \frac{x_1^2 + x_2^2 + 8}{10} \leq 1.25$$

and

$$\frac{8}{10} \le \frac{x_1x_2^2 + x_1 + 8}{10} \le 1.2875,$$

so $\mathbf{G}(\mathbf{x}) \in D$, whenever $\mathbf{x} \in D$.

Further,

$$\frac{\partial g_1}{\partial x_1} = \frac{2x_1}{10} \quad \text{so} \quad \left|\frac{\partial g_1(\mathbf{x})}{\partial x_1}\right| \le \frac{3}{10}, \quad \frac{\partial g_1}{\partial x_2} = \frac{2x_2}{10} \quad \text{so} \quad \left|\frac{\partial g_2(\mathbf{x})}{\partial x_2}\right| \le \frac{3}{10},$$

$$\frac{\partial g_2}{\partial x_1} = \frac{x_2^2+1}{10} \quad \text{so} \quad \left|\frac{\partial g_2(\mathbf{x})}{\partial x_1}\right| \le \frac{3.25}{10}, \quad \text{and} \quad \frac{\partial g_2}{\partial x_2} = \frac{2x_1x_2}{10} \quad \text{so} \quad \left|\frac{\partial g_2(\mathbf{x})}{\partial x_2}\right| \le \frac{4.5}{10}.$$

Since

$$\left|\frac{\partial g_i(\mathbf{x})}{\partial x_j}\right| \le 0.45 = \frac{0.9}{2},$$

for $i, j = 1, 2$, all hypothesis of Theorem 10.6 have been satisfied, and $\mathbf{G}$ has a unique fixed point in D.

(b) With $\mathbf{x}^{(0)} = (0,0)^t$ and tolerance 10^{-5}, we have $\mathbf{x}^{(13)} = (0.9999973, 0.9999973)^t$.

(c) With $\mathbf{x}^{(0)} = (0,0)^t$ and tolerance 10^{-5}, we have $\mathbf{x}^{(11)} = (0.9999984, 0.9999991)^t$.

6. (a) $\mathbf{G} = \left(x_2/\sqrt{5}, 0.25(\sin x_1 + \cos x_2)\right)^t$ and $D = \{(x_1, x_2) \mid 0 \le x_1, x_1 \le 1\}$.

(b) With $\mathbf{x}^{(0)} = \left(\frac{1}{2}, \frac{1}{2}\right)^t$, we have $\mathbf{x}^{(10)} = (0.1212440, 0.2711065)^t$.

(c) With $\mathbf{x}^{(0)} = \left(\frac{1}{2}, \frac{1}{2}\right)^t$, we have $\mathbf{x}^{(5)} = (0.1212421, 0.2711052)^t$.

7. (a) With $\mathbf{x}^{(0)} = (1,1,1)^t$, we have $\mathbf{x}^{(5)} = (5.0000000, 0.0000000, -0.5235988)^t$.

(b) With $\mathbf{x}^{(0)} = (1,1,1)^t$, we have $\mathbf{x}^{(9)} = (1.0364011, 1.0857072, 0.93119113)^t$.

(c) With $\mathbf{x}^{(0)} = (0,0,0.5)^t$, we have $\mathbf{x}^{(5)} = (0.00000000, 0.09999999, 1.0000000)^t$.

(d) With $\mathbf{x}^{(0)} = (0,0,0)^t$, we have $\mathbf{x}^{(5)} = (0.49814471, -0.19960600, -0.52882595)^t$.

8. (a) With

$$\mathbf{G}(\mathbf{x}) = \left(\sqrt{x_1 - x_2^2}, \sqrt{x_1^2 - x_2}\right)^t \quad \text{and} \quad \mathbf{x}^{(0)} = (0.7, 0.4)^t,$$

we have $\mathbf{x}^{(14)} = (0.77184647, 0.41965131)^t$.

(b) With

$$\mathbf{G}(\mathbf{x}) = \left(x/\sqrt{3}, \sqrt{(1+x_1^3)/(3x_1)}\right)^t \quad \text{and} \quad \mathbf{x}^{(0)} = (0.4, 0.7)^t,$$

we have $\mathbf{x}^{(20)} = (0.4999980, 0.8660221)^t$.

(c) With

$$\mathbf{G}(\mathbf{x}) = \left(\sqrt{37 - x_2}, \sqrt{x_1 - 5}, 3 - x_1 - x_2\right)^t \quad \text{and} \quad \mathbf{x}^{(0)} = (5, 1, -1)^t,$$

we have $\mathbf{x}^{(10)} = (6.0000002, 1.0000000, -3.9999971)^t$.

(d) With

$$\mathbf{G}(\mathbf{x}) = \left(\sqrt{2x_3 + x_2 - 2x_2^2}, \sqrt{(10x_3 + x_1^2)/8}, x_1^2/(7x_2)\right)^t \quad \text{and} \quad \mathbf{x}^{(0)} = (0.5, 0.5, 0)^t,$$

we have $\mathbf{x}^{(60)} = (0.5291548, 0.4000018, 0.09999853)^t$.

9. (a) With $\mathbf{x}^{(0)} = (1,1,1)^t$, we have $\mathbf{x}^{(3)} = (0.5000000, 0, -0.5235988)^t$.
(b) With $\mathbf{x}^{(0)} = (1,1,1)^t$, we have $\mathbf{x}^{(4)} = (1.036400, 1.085707, 0.9311914)^t$.
(c) With $\mathbf{x}^{(0)} = (0,0,0)^t$, we have $\mathbf{x}^{(3)} = (0, 0.1000000, 1.0000000)^t$.
(d) With $\mathbf{x}^{(0)} = (0,0,0)^t$, we have $\mathbf{x}^{(4)} = (0.4981447, -0.1996059, -0.5288260)^t$.

10. (a) Using $\mathbf{G}_1(\mathbf{x}) = (\sqrt{x_1 - x_2{}^2}, \sqrt{x_1{}^2 - x_2})^t$ and $\mathbf{x}^{(0)} = (0.7, 0.4)^t$ as in Exercise 8(a) gives a square root of a negative number as the first iteration. Thus, the method fails.
(b) Using $\mathbf{G}_1(\mathbf{x}) = \left(x/\sqrt{3}, \sqrt{(1+x_1{}^3)/(3x_1)}\right)^t$ and $\mathbf{x}^{(0)} = (0.4, 0.7)^t$ as in Exercise 8(b) gives $\mathbf{x}^{(10)} = (0.49999807, 0.86602652)^t$. The convergence is accelerated for this problem.
(c) Using $\mathbf{G}_1(\mathbf{x}) = (\sqrt{37 - x_2}, \sqrt{x_1 - 5}, 3 - x_1 - x_2)^t$ and $\mathbf{x}^{(0)} = (5, 1, -1)^t$ as in Exercise 8(c) gives $\mathbf{x}^{(1)} = (6, 1, -4)^t$. The convergence very much accelerated for this problem.
(d) Using $\mathbf{G}_1(\mathbf{x}) = (\sqrt{2x_3 + x_2 - 2x_2{}^2}, \sqrt{(10x_3 + x_1{}^2)/8}, x_1{}^2/(2x_2))^t$ and $\mathbf{x}^{(0)} = (0.5, 0.5, 0)^t$ as in Exercise 8(d) leads to division by zero as the first iteration. Thus, the method fails.

11. A stable solution occurs when $x_1 = 8000$ and $x_2 = 4000$.

12. Let $\mathbf{F}(\mathbf{x}) = (f_1(\mathbf{x}), ..., f_n(\mathbf{x}))^t$. Suppose $\mathbf{F}$ is continuous at $\mathbf{x_0}$. By Definition 10.3,

$$\lim_{\mathbf{x} \to \mathbf{x_0}} f_i(\mathbf{x}) = f_i(\mathbf{x_0}), \quad \text{for each } i = 1, ..., n.$$

Given $\epsilon > 0$, there exists $\delta_i > 0$ such that

$$|f_i(\mathbf{x}) - f_i(\mathbf{x_0})| < \epsilon,$$

whenever $0 < ||\mathbf{x} - \mathbf{x_0}|| < \delta_i$ and $\mathbf{x} \in D$.

Let $\delta = \min_{1 \le i \le n} \delta_i$. If $0 < ||\mathbf{x} - \mathbf{x_0}|| < \delta$, then $0 < ||\mathbf{x} - \mathbf{x_0}|| < \delta_i$ and $|f_i(\mathbf{x}) - f_i(\mathbf{x_0})| < \epsilon$, for each $i = 1, ..., n$, whenever $\mathbf{x} \in D$. This implies that

$$||\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{x_0})||_\infty < \epsilon,$$

whenever $||\mathbf{x} - \mathbf{x_0}|| < \delta$ and $\mathbf{x} \in D$. By the equivalence of vector norms, the result holds for all vector norms by suitably adjusting δ.

For the converse, let $\epsilon > 0$ be given. Then there is a $\delta > 0$ such that

$$||\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{x_0})|| < \epsilon,$$

whenever $\mathbf{x} \in D$ and $||\mathbf{x} - \mathbf{x_0}|| < \delta$. By the equivalence of vector norms, a number $\delta' > 0$ can be found with

$$||f_i(\mathbf{x}) - f_i(\mathbf{x_0})|| < \epsilon,$$

whenever $\mathbf{x} \in D$ and $||\mathbf{x} - \mathbf{x_0}|| < \delta'$.

Thus, $\lim_{\mathbf{x} \to \mathbf{x_0}} f_i(\mathbf{x}) = f_i(\mathbf{x_0})$, for $i = 1, ..., n$. Since $\mathbf{F}(\mathbf{x_0})$ is defined, the conditions in Definition 10.3 hold, and $\mathbf{F}$ is continuous at $\mathbf{x_0}$.

Exercise Set 10.2, page 613

1. Newton's method gives the following:

   (a) $\mathbf{x}^{(2)} = (0.4958936, 1.983423)^t$

   (b) $\mathbf{x}^{(2)} = (-0.5131616, -0.01837622)^t$

   (c) $\mathbf{x}^{(2)} = (-23.942626, 7.6086797)^t$

   (d) $\mathbf{x}^{(1)}$ cannot be computed since $J(0)$ is singular.

2. Newton's method gives the following:

   (a) $\mathbf{x}^{(2)} = (0.5001667, 0.2508036, -0.5173874)^t$

   (b) $\mathbf{x}^{(2)} = (4.350877, 18.49123, -19.84211)^t$

   (c) $\mathbf{x}^{(2)} = (1.03668708, 1.08592384, 0.92977932)^t$

   (d) $\mathbf{x}^{(2)} = (0.40716687, 1.30944377, -0.85895477)^t$

3. Graphing in Maple gives the following:

   (a) $(0.5, 0.2)^t$ and $(1.1, 6.1)^t$

   (b) $(-0.35, 0.05)^t, (0.2, -0.45)^t, (0.4, -0.5)^t$ and $(1, -0.3)^t$

   (c) $(-1, 3.5)^t$, $(2.5, 4)^t$

   (d) $(0.11, 0.27)^t$

4. Graphing in Maple gives the following:

   (a) $(0.5, 0.5, -0.5)^t$

   (b) $(7, -1, -2)^t$

   (c) $(1, 1, 1)^t$

   (d) $(1, -1, 1)^t$ and $(1, 1, -1)^t$

5. Newton's method gives the following:

   (a) With $\mathbf{x}^{(0)} = (0.5, 2)^t, \mathbf{x}^{(3)} = (0.5, 2)^t$ With $\mathbf{x}^{(0)} = (1.1, 6.1), \mathbf{x}^{(3)} = (1.0967197, 6.0409329)^t$

   (b) With $\mathbf{x}^{(0)} = (-0.35, 0.05)^t, \mathbf{x}^{(3)} = (-0.37369822, 0.056266490^t$ With $\mathbf{x}^{(0)} = (0.2, -0.45)^t, \mathbf{x}^{(4)} = (0.14783924, -0.43617762)^t$ With $\mathbf{x}^{(0)} = (0.4, -0.5)^t, \mathbf{x}^{(3)} = (0.40809566, -0.49262939)^t$ With $\mathbf{x}^{(0)} = (1, -0.3)^t, \mathbf{x}^{(4)} = (1.0330715, -0.27996184)^t$

   (c) With $\mathbf{x}^{(0)} = (-1, 3.5)^t$, $\mathbf{x}^{(1)} = (-1, 3.5)^t$ and $\mathbf{x}^{(0)} = (2.5, 4)^t$, $\mathbf{x}^{(3)} = (2.546947, 3.984998)^t$.

   (d) With $\mathbf{x}^{(0)} = (0.11, 0.27)^t$, $\mathbf{x}^{(6)} = (0.1212419, 0.2711051)^t$.

6. Newton's method gives the following:

   (a) $\mathbf{x}^{(12)} = (0.49999953, 0.00319906, -0.52351886)^t$

   (b) $\mathbf{x}^{(4)} = (6.17107462, -1.08216201, -2.08891251)^t$

   (c) With $\mathbf{x}^{(0)} = (1, 1, 1)^t$, $\mathbf{x}^{(3)} = (1.036401, 1.085707, 0.9311914)^t$.

   (d) With $\mathbf{x}^{(0)} = (1, -1, 1)^t$, $\mathbf{x}^{(5)} = (0.9, -1, 0.5)^t$; and with $\mathbf{x}^{(0)} = (1, -1, 1)^t$, $\mathbf{x}^{(5)} = (0.5, 1, -0.5)^t$.

7. Newton's method gives the following:

(a) $\mathbf{x}^{(5)} = (0.5000000, 0.8660254)^t$ (b) $\mathbf{x}^{(6)} = (1.772454, 1.772454)^t$

(c) $\mathbf{x}^{(5)} = (-1.456043, -1.664230, 0.4224934)^t$

(d) $\mathbf{x}^{(4)} = (0.4981447, -0.1996059, -0.5288260)^t$

8. (a) Suppose $(x_1, x_2, x_3, x_4)^t$ is a solution to

$$\begin{aligned} 4x_1 - x_2 + x_3 &= x_1x_4, \\ -x_1 + 3x_2 - 2x_3 &= x_2x_4, \\ x_1 - 2x_2 + 3x_3 &= x_3x_4, \\ x_1^2 + x_2^2 + x_3^2 &= 1. \end{aligned}$$

Multiplying the first three equations by -1 and factoring gives

$$\begin{aligned} 4(-x_1) - (-x_2) + (-x_3) &= (-x_1)x_4, \\ -(-x_1) + 3(-x_2) - 2(-x_3) &= (-x_2)x_4, \\ (-x_1) - 2(-x_2) + 3(-x_3) &= (-x_3)x_4, \\ (-x_1)^2 + (-x_2)^2 + (-x_3)^2 &= 1. \end{aligned}$$

Thus, $(-x_1, -x_2, -x_3, x_4)^t$ is also a solution.

(b) Using $\mathbf{x}^{(0)} = (1, 1, 1, 1)^t$ gives $\mathbf{x}^{(5)} = (0, 0.70710678, 0.70710678, 1)^t$.
Using $\mathbf{x}^{(0)} = (1, 0, 0, 0)^t$ gives $\mathbf{x}^{(6)} = (0.81649658, 0.40824829, -0.40824829, 3)^t$.
Using $\mathbf{x}^{(0)} = (1, -1, 1, -1)^t$ gives $\mathbf{x}^{(5)} = (0.57735027, -0.57735027, 0.57735027, 6)^t$.
The other three solutions follow easily from part (a).

9. With $\mathbf{x}^{(0)} = (1, 1-1)^t$ and $TOL = 10^{-6}$, we have $\mathbf{x}^{(20)} = (0.5, 9.5 \times 10^{-7}, -0.5235988)^t$.

10. Since $f_j(x_1, \ldots, x_n) = a_{j1}x_1 + a_{j2}x_2 + \ldots + a_{jn}x_n - b_j$, we have $\dfrac{\partial f_j}{\partial x_1} = a_{ji}$. Hence,

$$J(\mathbf{x}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = A.$$

Further,

$$\begin{aligned} \mathbf{F}\left(\mathbf{x}^{(0)}\right) &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \\ x_n^{(0)} \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \\ &= J\left(\mathbf{x}^{(0)}\right)\mathbf{x}^{(0)} - \mathbf{b}. \end{aligned}$$

Thus, given $\mathbf{x}^{(0)}$, we have

$$\begin{aligned}\mathbf{x}^{(1)} &= \mathbf{x}^{(0)} - J\left(\mathbf{x}^{(0)}\right)^{-1}\left(J\left(\mathbf{x}^{(0)}\right)\mathbf{x}^{(0)} - \mathbf{b}\right) \\ &= \mathbf{x}^{(0)} - J\left(\mathbf{x}^{(0)}\right)^{-1} J\left(\mathbf{x}^{(0)}\right)\mathbf{x}^{(0)} + J\left(\mathbf{x}^{(0)}\right)^{-1}\mathbf{b} \\ &= J\left(\mathbf{x}^{(0)}\right)^{-1}\mathbf{b} = A^{-1}\mathbf{b}.\end{aligned}$$

So given any $\mathbf{x}^{(0)}$, the solution to the linear system is $\mathbf{x}^{(1)}$.

11. When the dimension n is 1, $\mathbf{F}(\mathbf{x})$ is a one-component function $f(\mathbf{x}) = f_1(\mathbf{x})$, and the vector $\mathbf{x}$ has only one component $x_1 = x$. In this case, the Jacobian matrix $J(\mathbf{x})$ reduces to the 1×1 matrix $[\partial f_1/\partial x_1(\mathbf{x})] = f'(\mathbf{x}) = f'(x)$. Thus, the vector equation

$$\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} - J(\mathbf{x}^{(k-1)})^{-1}\mathbf{F}(\mathbf{x}^{(k-1)})$$

becomes the scalar equation

$$x_k = x_{k-1} - f(x_{k-1})^{-1} f(x_{k-1}) = x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})}.$$

12. The constants required for the pressure equation are in part (a). The approximate radius is in part (b).

(a) $k_1 = 8.77125, k_2 = 0.259690, k_3 = -1.37217$

(b) Solving the equation

$$\frac{500}{\pi r^2} = k_1 e^{k_2 r} + k_3 r$$

numerically, gives $r = 3.18517$.

13. With $\theta_i^{(0)} = 1$, for each $i = 1, 2, \ldots, 20$, the following results are obtained.

i	1	2	3	4	5	6
$\theta_i^{(5)}$	0.14062	0.19954	0.24522	0.28413	0.31878	0.35045

i	7	8	9	10	11	12	13
$\theta_i^{(5)}$	0.37990	0.40763	0.43398	0.45920	0.48348	0.50697	0.52980

i	14	15	16	17	18	19	20
$\theta_i^{(5)}$	0.55205	0.57382	0.59516	0.61615	0.63683	0.65726	0.67746