HW16, 2

Let $F = \begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0.03\sin(x_1+x_2)-4 \\ 0.07\cos(x_1-x_2)-8 \end{pmatrix}$

if $\exists$ a invertible matrix $\alpha_{2\times2}$ $\ni$

$$\alpha_{2\times2} F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \alpha_{2\times2}\begin{pmatrix} 0.03\sin(x_1+x_2)-4 \\ 0.07\cos(x_1-x_2)-8 \end{pmatrix}$$

$$= F'$$

then $F' = \vec{0} \Leftrightarrow F = \vec{0}$ .... (1)

Let $G = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - F'$, then if $\exists$ fixed point $\vec{x}^*$

$\ni$ $\vec{x}^* = G(\vec{x}^*) = \vec{x}^* - F'(\vec{x}^*)$

$\Rightarrow$ $F'(\vec{x}^*) = \vec{0} \Rightarrow F = \vec{0}$ .... (2)

try to check $\left|\frac{\partial}{\partial x_i} G_j\right| < \frac{1}{2}$, for $1 \le i, j \le 2$

Let $\alpha_{2\times2} = \begin{pmatrix} -3/2 & 1/2 \\ 5/4 & -1/4 \end{pmatrix}$

$$\Rightarrow G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2}(0.03\cdot\sin(x_1+x_2)-4) - \frac{1}{2}(0.07\cdot\cos(x_1-x_2)-8) \\ -\frac{5}{4}(0.03\cdot\sin(x_1+x_2)-4) + \frac{1}{4}(0.07\cdot\cos(x_1-x_2)-8) \end{pmatrix}$$

$$\frac{\partial G_1}{\partial x_1} = \frac{0.09}{2}\cdot\cos(x_1+x_2) + \frac{0.07}{2}\sin(x_1-x_2)$$

$$\frac{\partial G_1}{\partial x_2} = \frac{0.09}{2}\cdot\cos(x_1+x_2) - \frac{0.07}{2}\sin(x_1-x_2)$$

$$\frac{\partial G_2}{\partial x_1} = \frac{0.15}{4}\cos(x_1+x_2) - \frac{0.07}{4}\sin(x_1-x_2)$$

$$\frac{\partial G_2}{\partial x_2} = \frac{0.15}{4}\cos(x_1+x_2) + \frac{0.07}{4}\sin(x_1-x_2)$$

clearly, $\left|\frac{\partial G_j}{\partial x_i}\right| < 0.5$ for $1 \le i, j \le 2$

Hence, by Theorem 10.6, The sequence $\{x^{(k)}\}_{k=0}^{\infty}$ defined by an arbitrarily selected $\vec{x}^{(0)}$ in R^2 and generated by $\vec{x}^{(k)} = G(\vec{x}^{(k-1)})$, for each $k \ge 1$ converges to the unique fixed point $p \in R^2$ and

$$\| x^{(k)} - p \|_\infty \le \frac{K^k}{1-K} \| \vec{x}^{(1)} - \vec{x}^{(0)} \|_\infty$$

Note that $K \approx \max\left\{ 0.09 + 0.07,\ \frac{0.15}{2} + \frac{0.07}{2} \right\}$

$= 0.16$ "$\left|\frac{\partial G_j}{\partial x_i}\right| < \frac{K}{2}$" #