24. Since g' is continuous at p and $|g'(p)| > 1$, by letting $\epsilon = |g'(p)| - 1$ there exists a number $\delta > 0$ such that $|g'(x) - g'(p)| < |g'(p)| - 1$ whenever $0 < |x - p| < \delta$. Hence, for any x satisfying $0 < |x - p| < \delta$, we have

$$|g'(x)| \geq |g'(p)| - |g'(x) - g'(p)| > |g'(p)| - (|g'(p)| - 1) = 1.$$

If p_0 is chosen so that $0 < |p - p_0| < \delta$, we have by the Mean Value Theorem that

$$|p_1 - p| = |g(p_0) - g(p)| = |g'(\xi)||p_0 - p|,$$

for some ξ between p_0 and p. Thus, $0 < |p - \xi| < \delta$ so $|p_1 - p| = |g'(\xi)||p_0 - p| > |p_0 - p|$.

Exercise Set 2.3, page 71

1. $p_2 = 2.60714$

2. $p_2 = -0.865684$; If $p_0 = 0$, $f'(p_0) = 0$ and p_1 cannot be computed.

3. (a) 2.45454 (b) 2.44444 (c) Part (a) is better.

4. (a) -1.25208 (b) -0.841355

5. (a) For $p_0 = 2$, we have $p_5 = 2.69065$.
   (b) For $p_0 = -3$, we have $p_3 = -2.87939$.
   (c) For $p_0 = 0$, we have $p_4 = 0.73909$.
   (d) For $p_0 = 0$, we have $p_3 = 0.96434$.

6. (a) For $p_0 = 1$, we have $p_8 = 1.829384$.
   (b) For $p_0 = 1.5$, we have $p_4 = 1.397748$.
   (c) For $p_0 = 2$, we have $p_4 = 2.370687$; and for $p_0 = 4$, we have $p_4 = 3.722113$.
   (d) For $p_0 = 1$, we have $p_4 = 1.412391$; and for $p_0 = 4$, we have $p_5 = 3.057104$.
   (e) For $p_0 = 1$, we have $p_4 = 0.910008$; and for $p_0 = 3$, we have $p_9 = 3.733079$.
   (f) For $p_0 = 0$, we have $p_4 = 0.588533$; for $p_0 = 3$, we have $p_3 = 3.096364$; and for $p_0 = 6$, we have $p_3 = 6.285049$.

7. Using the endpoints of the intervals as p_0 and p_1, we have:

   (a) $p_{11} = 2.69065$ (b) $p_7 = -2.87939$ (c) $p_6 = 0.73909$ (d) $p_5 = 0.96433$

8. Using the endpoints of the intervals as p_0 and p_1, we have:

   (a) $p_7 = 1.829384$ (b) $p_9 = 1.397749$

(c) $p_6 = 2.370687; p_7 = 3.722113$ (d) $p_8 = 1.412391; p_7 = 3.057104$
(e) $p_6 = 0.910008; p_{10} = 3.733079$
(f) $p_6 = 0.588533; p_5 = 3.096364; p_5 = 6.285049$

9. Using the endpoints of the intervals as p_0 and p_1, we have:

(a) $p_{16} = 2.69060$ (b) $p_6 = -2.87938$ (c) $p_7 = 0.73908$ (d) $p_6 = 0.96433$

10. Using the endpoints of the intervals as p_0 and p_1, we have:

(a) $p_8 = 1.829383$ (b) $p_9 = 1.397749$
(c) $p_6 = 2.370687; p_8 = 3.722112$ (d) $p_{10} = 1.412392; p_{12} = 3.057099$
(e) $p_7 = 0.910008; p_{29} = 3.733065$
(f) $p_9 = 0.588533; p_5 = 3.096364; p_5 = 6.285049$

11. (a) Newton's method with $p_0 = 1.5$ gives $p_3 = 1.51213455$.
The Secant method with $p_0 = 1$ and $p_1 = 2$ gives $p_{10} = 1.51213455$.
The Method of False Position with $p_0 = 1$ and $p_1 = 2$ gives $p_{17} = 1.51212954$.
(b) Newton's method with $p_0 = 0.5$ gives $p_5 = 0.976773017$.
The Secant method with $p_0 = 0$ and $p_1 = 1$ gives $p_5 = 10.976773017$.
The Method of False Position with $p_0 = 0$ and $p_1 = 1$ gives $p_5 = 0.976772976$.

12. (a)

	Initial Approximation	Result	Initial Approximation	Result
Newton's	$p_0 = 1.5$	$p_4 = 1.41239117$	$p_0 = 3.0$	$p_4 = 3.05710355$
Secant	$p_0 = 1, p_1 = 2$	$p_8 = 1.41239117$	$p_0 = 2, p_1 = 4$	$p_{10} = 3.05710355$
False Position	$p_0 = 1, p_1 = 2$	$p_{13} = 1.41239119$	$p_0 = 2, p_1 = 4$	$p_{19} = 3.05710353$

(b)

	Initial Approximation	Result	Initial Approximation	Result
Newton's	$p_0 = 0.25$	$p_4 = 0.206035120$	$p_0 = 0.75$	$p_4 = 0.681974809$
Secant	$p_0 = 0, p_1 = 0.5$	$p_9 = 0.206035120$	$p_0 = 0.5, p_1 = 1$	$p_8 = 0.681974809$
False Position	$p_0 = 0, p_1 = 0.5$	$p_{12} = 0.206035125$	$p_0 = 0.5, p_1 = 1$	$p_{15} = 0.681974791$

13. For $p_0 = 1$, we have $p_5 = 0.589755$. The point has the coordinates $(0.589755, 0.347811)$.

14. For $p_0 = 2$, we have $p_2 = 1.866760$. The point is $(1.866760, 0.535687)$.

15. The equation of the tangent line is

$$y - f(p_{n-1}) = f'(p_{n-1})(x - p_{n-1}).$$

To complete this problem, set $y = 0$ and solve for $x = p_n$.

16. Newton's method gives $p_{15} = 1.895488$, for $p_0 = \frac{\pi}{2}$; and $p_{19} = 1.895489$, for $p_0 = 5\pi$. The sequence does not converge in 200 iterations for $p_0 = 10\pi$. The results do not indicate the fast convergence usually associated with Newton's method.

17. (a) For $p_0 = -1$ and $p_1 = 0$, we have $p_{17} = -0.04065850$, and for $p_0 = 0$ and $p_1 = 1$, we have $p_9 = 0.9623984$.

    (b) For $p_0 = -1$ and $p_1 = 0$, we have $p_5 = -0.04065929$, and for $p_0 = 0$ and $p_1 = 1$, we have $p_{12} = -0.04065929$.

    (c) For $p_0 = -0.5$, we have $p_5 = -0.04065929$, and for $p_0 = 0.5$, we have $p_{21} = 0.9623989$.

18. (a) The Bisection method yields $p_{10} = 0.4476563$.

    (b) The method of False Position yields $p_{10} = 0.442067$.

    (c) The Secant method yields $p_{10} = -195.8950$.

19. This formula involves the subtraction of nearly equal numbers in both the numerator and denominator if p_{n-1} and p_{n-2} are nearly equal.

20. Newton's method for the various values of p_0 gives the following results.

    (a) $p_8 = -1.379365$ (b) $p_7 = -1.379365$ (c) $p_7 = 1.379365$

    (d) $p_7 = -1.379365$ (e) $p_7 = 1.379365$ (f) $p_8 = 1.379365$

21. Newton's method for the various values of p_0 gives the following results.

    (a) $p_0 = -10, p_{11} = -4.30624527$
    (b) $p_0 = -5, p_5 = -4.30624527$
    (c) $p_0 = -3, p_5 = 0.824498585$
    (d) $p_0 = -1, p_4 = -0.824498585$
    (e) $p_0 = 0$, p_1 cannot be computed since $f'(0) = 0$
    (f) $p_0 = 1, p_4 = 0.824498585$
    (g) $p_0 = 3, p_5 = -0.824498585$
    (h) $p_0 = 5, p_5 = 4.30624527$
    (i) $p_0 = 10, p_{11} = 4.30624527$

22. The required accuracy is met in 7 iterations of Newton's method.

23. For $f(x) = \ln(x^2 + 1) - e^{0.4x} \cos \pi x$, we have the following roots.

    (a) For $p_0 = -0.5$, we have $p_3 = -0.4341431$.

    (b) For $p_0 = 0.5$, we have $p_3 = 0.4506567$.
    For $p_0 = 1.5$, we have $p_3 = 1.7447381$.
    For $p_0 = 2.5$, we have $p_5 = 2.2383198$.
    For $p_0 = 3.5$, we have $p_4 = 3.7090412$.

    (c) The initial approximation $n - 0.5$ is quite reasonable.

    (d) For $p_0 = 24.5$, we have $p_2 = 24.4998870$.

24. We have $\lambda \approx 0.100998$ and $N(2) \approx 2{,}187{,}950$.

25. The two numbers are approximately 6.512849 and 13.487151.

26. The minimal annual interest rate is 6.67%.

27. The borrower can afford to pay at most 8.10%.

28. (a) $\frac{1}{3}e, t = 3$ hours (b) 11 hours and 5 minutes (c) 21 hours and 14 minutes

29. (a) `solve(3^(3*x+1)-7*5^(2*x),x)` and `fsolve(3^(3*x+1)-7*5^(2*x),x)` both fail.
    (b) `plot(3^(3*x+1)-7*5^(2*x),x=a..b)` generally yields no useful information. However, $a = 10.5$ and $b = 11.5$ in the plot command show that $f(x)$ has a root near $x = 11$.
    (c) With $p_0 = 11$, $p_5 = 11.0094386442681716$ is accurate to 10^{-16}.
    (d) $p = \dfrac{\ln(3/7)}{\ln(25/27)}$

30. (a) `solve(2^(x^2)-3*7^(x+1),x)` fails and `fsolve(2^(x^2)-3*7^(x+1),x)` returns -1.118747530.
    (b) `plot(2^(x^2)-3*7^(x+1),x=-2..4)` shows there is also a root near $x = 4$.
    (c) With $p_0 = 1$, $p_4 = -1.1187475303988963$ is accurate to 10^{-16}; with $p_0 = 4$, $p_6 = 3.9261024524565005$ is accurate to 10^{-16}
    (d) The roots are
    $$\frac{\ln(7) \pm \sqrt{[\ln(7)]^2 + 4\ln(2)\ln(4)}}{2\ln(2)}.$$

31. We have $P_L = 265816$, $c = -0.75658125$, and $k = 0.045017502$. The 1980 population is $P(30) = 222{,}248{,}320$, and the 2010 population is $P(60) = 252{,}967{,}030$.

32. $P_L = 290228$, $c = 0.6512299$, and $k = 0.03020028$;
    The 1980 population is $P(30) = 223,069,210$, and the 2010 population is $P(60) = 260,943,806$.

33. Using $p_0 = 0.5$ and $p_1 = 0.9$, the Secant method gives $p_5 = 0.842$.

34. (b) Newton's method gives $\alpha \approx 33.2°$.

Exercise Set 2.4, page 82

1. (a) For $p_0 = 0.5$, we have $p_{13} = 0.567135$.
   (b) For $p_0 = -1.5$, we have $p_{23} = -1.414325$.
   (c) For $p_0 = 0.5$, we have $p_{22} = 0.641166$.
   (d) For $p_0 = -0.5$, we have $p_{23} = -0.183274$.

2. (a) For $p_0 = 0.5$, we have $p_{15} = 0.739076589$.
   (b) For $p_0 = -2.5$, we have $p_9 = -1.33434594$.
   (c) For $p_0 = 3.5$, we have $p_5 = 3.14156793$.

(d) For $p_0 = 4.0$, we have $p_{44} = 3.37354190$.

3. Modified Newton's method in Eq. (2.11) gives the following:

 (a) For $p_0 = 0.5$, we have $p_3 = 0.567143$.

 (b) For $p_0 = -1.5$, we have $p_2 = -1.414158$.

 (c) For $p_0 = 0.5$, we have $p_3 = 0.641274$.

 (d) For $p_0 = -0.5$, we have $p_5 = -0.183319$.

4. (a) For $p_0 = 0.5$, we have $p_4 = 0.739087439$.

 (b) For $p_0 = -2.5$, we have $p_{53} = -1.33434594$.

 (c) For $p_0 = 3.5$, we have $p_5 = 3.14156793$.

 (d) For $p_0 = 4.0$, we have $p_3 = -3.72957639$.

5. Newton's method with $p_0 = -0.5$ gives $p_{13} = -0.169607$. Modified Newton's method in Eq. (2.11) with $p_0 = -0.5$ gives $p_{11} = -0.169607$.

6. (a) Since

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1,$$

 we have linear convergence. To have $|p_n - p| < 5 \times 10^{-2}$, we need $n \geq 20$.

 (b) Since

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1,$$

 we have linear convergence. To have $|p_n - p| < 5 \times 10^{-2}$, we need $n \geq 5$.

7. (a) For $k > 0$,

$$\lim_{n\to\infty} \frac{|p_{n+1}-0|}{|p_n-0|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^k}}{\frac{1}{n^k}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^k = 1,$$

 so the convergence is linear.

 (b) We need to have $N > 10^{m/k}$.

8. (a) Since

$$\lim_{n\to\infty} \frac{|p_{n+1}-0|}{|p_n-0|^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{\left(10^{-2^n}\right)^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-2^{n+1}}} = 1,$$

 the sequence is quadratically convergent.

 (b) We have

$$\begin{aligned}\lim_{n\to\infty} \frac{|p_{n+1}-0|}{|p_n-0|^2} &= \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{\left(10^{-n^k}\right)^2} = \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{10^{-2n^k}} \\ &= \lim_{n\to\infty} 10^{2n^k-(n+1)^k} = \lim_{n\to\infty} 10^{n^k\left(2-\left(\frac{n+1}{n}\right)^k\right)} = \infty,\end{aligned}$$

 so the sequence $p_n = 10^{-n^k}$ does not converge quadratically.

9. Typical examples are

(a) $p_n = 10^{-3^n}$ (b) $p_n = 10^{-\alpha^n}$

10. Suppose $f(x) = (x-p)^m q(x)$. Since

$$g(x) = x - \frac{m(x-p)q(x)}{mq(x) + (x-p)q'(x)},$$

we have $g'(p) = 0$.

11. This follows from the fact that

$$\lim_{n\to\infty} \frac{\left|\frac{b-a}{2^{n+1}}\right|}{\left|\frac{b-a}{2^n}\right|} = \frac{1}{2}.$$

12. If f has a zero of multiplicity m at p, then f can be written as

$$f(x) = (x-p)^m q(x),$$

for $x \neq p$, where

$$\lim_{x\to p} q(x) \neq 0.$$

Thus,

$$f'(x) = m(x-p)^{m-1}q(x) + (x-p)^m q'(x)$$

and $f'(p) = 0$. Also,

$$f''(x) = m(m-1)(x-p)^{m-2}q(x) + 2m(x-p)^{m-1}q'(x) + (x-p)^m q''(x)$$

and $f''(p) = 0$. In general, for $k \leq m$,

$$\begin{aligned} f^{(k)}(x) =& \sum_{j=0}^{k} \binom{k}{j} \frac{d^j(x-p)^m}{dx^j} q^{(k-j)}(x) \\ =& \sum_{j=0}^{k} \binom{k}{j} m(m-1)\cdots(m-j+1)(x-p)^{m-j} q^{(k-j)}(x). \end{aligned}$$

Thus, for $0 \leq k \leq m-1$, we have $f^{(k)}(p) = 0$, but

$$f^{(m)}(p) = m! \lim_{x\to p} q(x) \neq 0.$$

Conversely, suppose that $f(p) = f'(p) = \ldots = f^{(m-1)}(p) = 0$ and $f^{(m)}(p) \neq 0$. Consider the $(m-1)$th Taylor polynomial of f expanded about p:

$$\begin{aligned} f(x) =& f(p) + f'(p)(x-p) + \ldots + \frac{f^{(m-1)}(p)(x-p)^{m-1}}{(m-1)!} + \frac{f^{(m)}(\xi(x))(x-p)^m}{m!} \\ =& (x-p)^m \frac{f^{(m)}(\xi(x))}{m!}, \end{aligned}$$

where $\xi(x)$ is between x and p. Since $f^{(m)}$ is continuous, let

$$q(x) = \frac{f^{(m)}(\xi(x))}{m!}.$$

Then $f(x) = (x-p)^m q(x)$ and

$$\lim_{x\to p} q(x) = \frac{f^{(m)}(p)}{m!} \neq 0.$$

13. If

$$\frac{|p_{n+1}-p|}{|p_n-p|^3} = 0.75 \quad \text{and} \quad |p_0-p| = 0.5,$$

then

$$|p_n - p| = (0.75)^{(3^n-1)/2}|p_0-p|^{3^n}.$$

To have $|p_n - p| \le 10^{-8}$ requires that $n \ge 3$.

14. Let $e_n = p_n - p$. If

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^\alpha} = \lambda > 0,$$

then for sufficiently large values of n, $|e_{n+1}| \approx \lambda |e_n|^\alpha$. Thus,

$$|e_n| \approx \lambda |e_{n-1}|^\alpha \quad \text{and} \quad |e_{n-1}| \approx \lambda^{-1/\alpha}|e_n|^{1/\alpha}.$$

Using the hypothesis gives

$$\lambda |e_n|^\alpha \approx |e_{n+1}| \approx C|e_n|\lambda^{-1/\alpha}|e_n|^{1/\alpha},$$

so

$$|e_n|^\alpha \approx C\lambda^{-1/\alpha-1}|e_n|^{1+1/\alpha}.$$

Since the powers of $|e_n|$ must agree,

$$\alpha = 1 + 1/\alpha \;\text{ and }\; \alpha = \frac{1+\sqrt{5}}{2} \approx 1.62.$$

The number α is the *golden ratio* that appeared in Exercise 16 of section 1.3.

Exercise Set 2.5, page 86

1. The results are listed in the following table.

	(a)	(b)	(c)	(d)
$\hat{p}_0$	0.258684	0.907859	0.548101	0.731385
$\hat{p}_1$	0.257613	0.909568	0.547915	0.736087
$\hat{p}_2$	0.257536	0.909917	0.547847	0.737653
$\hat{p}_3$	0.257531	0.909989	0.547823	0.738469
$\hat{p}_4$	0.257530	0.910004	0.547814	0.738798
$\hat{p}_5$	0.257530	0.910007	0.547810	0.738958

2. Newton's Method gives $p_6 = -0.1828876$ and $\hat{p}_6 = -0.183387$.

3. Steffensen's method gives $p_0^{(1)} = 0.826427$.

4. Steffensen's method gives $p_0^{(1)} = 2.152905$ and $p_0^{(2)} = 1.873464$.

5. Steffensen's method gives $p_1^{(0)} = 1.5$.

6. Steffensen's method gives $p_2^{(0)} = 1.73205$.

7. For $g(x) = \sqrt{1 + \frac{1}{x}}$ and $p_0 = 1$, we have $p_3 = 1.32472$.

8. For $g(x) = 2^{-x}$ and $p_0 = 1$, we have $p_3 = 0.64119$.

9. For $g(x) = 0.5(x + \frac{3}{x})$ and $p_0 = 0.5$, we have $p_4 = 1.73205$.

10. For $g(x) = \frac{5}{\sqrt{x}}$ and $p_0 = 2.5$, we have $p_3 = 2.92401774$.

11. (a) For $g(x) = \left(2 - e^x + x^2\right)/3$ and $p_0 = 0$, we have $p_3 = 0.257530$.
    (b) For $g(x) = 0.5(\sin x + \cos x)$ and $p_0 = 0$, we have $p_4 = 0.704812$.
    (c) With $p_0 = 0.25$, $p_4 = 0.910007572$.
    (d) With $p_0 = 0.3$, $p_4 = 0.469621923$.

12. (a) For $g(x) = 2 + \sin x$ and $p_0 = 2$, we have $p_4 = 2.55419595$.
    (b) For $g(x) = \sqrt[3]{2x + 5}$ and $p_0 = 2$, we have $p_2 = 2.09455148$.
    (c) With $g(x) = \sqrt{\frac{e^x}{3}}$ and $p_0 = 1$, we have $p_3 = 0.910007574$.
    (d) With $g(x) = \cos x$, and $p_0 = 0$, we have $p_4 = 0.739085133$.

13. Aitken's Δ^2 method gives:

    (a) $\hat{p}_{10} = 0.0\overline{45}$ (b) $\hat{p}_2 = 0.0363$

14. (a) A positive constant λ exists with

$$\lambda = \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha}.$$

Hence,

$$\lim_{n\to\infty} \left| \frac{p_{n+1} - p}{p_n - p} \right| = \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} \cdot |p_n - p|^{\alpha - 1} = \lambda \cdot 0 = 0$$

and

$$\lim_{n\to\infty} \frac{p_{n+1} - p}{p_n - p} = 0.$$

(b) One example is $p_n = \frac{1}{n^n}$.

15. We have

$$\frac{|p_{n+1}-p_n|}{|p_n-p|} = \frac{|p_{n+1}-p+p-p_n|}{|p_n-p|} = \left|\frac{p_{n+1}-p}{p_n-p} - 1\right|,$$

so

$$\lim_{n\to\infty} \frac{|p_{n+1}-p_n|}{|p_n-p|} = \lim_{n\to\infty} \left|\frac{p_{n+1}-p}{p_n-p} - 1\right| = 1.$$

16.

$$\frac{\hat{p}_n - p}{p_n - p} = \frac{\lambda(\delta_n+\delta_{n+1}) - 2\delta_n + \delta_n\delta_{n+1} - 2\delta_n(\lambda-1) - \delta_n^2}{(\lambda-1)^2 + \lambda(\delta_n+\delta_{n+1}) - 2\delta_n + \delta_n\delta_{n+1}}$$

17. (a) First use the Taylor series for e^x to show that

$$p_n - p = -\frac{1}{(n+1)!}e^{\xi}x^{n+1},$$

where ξ is between 0 and 1. This implies that for large values of n we have

$$\left|\frac{p_{n+1}-p}{p_n-p}\right| = \left|\frac{e^{(\xi_1-\xi_2)}}{n+2}\,x\right| \le 1.$$

(b)

n	p_n	$\hat{p}_n$
0	1	3
1	2	2.75
2	2.5	$2.7\overline{2}$
3	$2.\overline{6}$	2.71875
4	$2.708\overline{3}$	$2.718\overline{3}$
5	$2.71\overline{6}$	2.7182870
6	$2.7180\overline{5}$	2.7182823
7	2.7182539	2.7182818
8	2.7182787	2.7182818
9	2.7182815	
10	2.7182818	

Exercise Set 2.6, page 96

1. (a) For $p_0 = 1$, we have $p_{22} = 2.69065$.
   (b) For $p_0 = 1$, we have $p_5 = 0.53209$; for $p_0 = -1$, we have $p_3 = -0.65270$; and for $p_0 = -3$, we have $p_3 = -2.87939$.
   (c) For $p_0 = 1$, we have $p_5 = 1.32472$.
   (d) For $p_0 = 1$, we have $p_4 = 1.12412$; and for $p_0 = 0$, we have $p_8 = -0.87605$.
   (e) For $p_0 = 0$, we have $p_6 = -0.47006$; for $p_0 = -1$, we have $p_4 = -0.88533$; and for $p_0 = -3$, we have $p_4 = -2.64561$.