HW 15.

3. $x_{n+1} = g(x_n) = g(x^* + \delta_n) = g(x^*) + \delta_n g'(x^*)$

$$+ \frac{\delta_n^2}{2!} g''(x^*) + \frac{\delta_n^3}{3!} g'''(\xi_n)$$

where $\delta_n = x_n - x^*$, ξ_n between x^* and x_n

$\because g'(x^*) = g''(x^*) = 0$

$\Rightarrow \quad x_{n+1} = g(x^*) + \frac{\delta_n^3}{3!} g'''(\xi_n)$

$\Rightarrow \quad |x_{n+1} - x^*| = \frac{\delta_n^3}{3!} g'''(\xi_n)$

then $\frac{|x_{n+1} - x^*|}{|x_n - x^*|^\alpha} = \frac{|\delta_n|^3}{|\delta_n|^\alpha} \cdot \frac{|g'''(\xi_n)|}{3!}$

$\because |g'(x)| \approx 0 < 1$ when x near x^*

$\therefore x_{n+1} = g(x_n)$ converges, when x_0 near x_n

$\therefore \delta_n \to 0$ when x_n converges.

$\Rightarrow \begin{cases} \text{if } \alpha < 3, \lim_{n\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^\alpha} \to 0 \\ \text{if } \alpha = 3, \lim_{n\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^\alpha} = \lim_{n\to\infty} \frac{|g'''(\xi_n)|}{3!} = \frac{|g'''(x^*)|}{3!} \end{cases}$

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HW15.

6. (i) $a_n = \frac{1}{n} \Rightarrow \hat{a}_n = a_n - \frac{(\Delta a_n)^2}{\Delta^2 a_n}$

$$= \frac{1}{n} - \frac{\left(\frac{1}{n+1} - \frac{1}{n}\right)^2}{\frac{1}{n+2} - \frac{2}{n+1} + \frac{1}{n}}$$

$$= \frac{1}{n} - \frac{\left(\frac{-1}{n(n+1)}\right)^2 (n+2)(n+1)(n)}{n(n+1) - 2n(n+2) + (n+1)(n+2)}$$

$$= \frac{1}{n} - \frac{\frac{n+2}{n(n+1)}}{2} = \frac{1}{2(n+1)}$$

clearly, a_n converges to 0 linearly

and $\lim_{n\to\infty} \frac{|\hat{a}_{n+1} - 0|}{|\hat{a}_n - 0|} = \lim_{n\to\infty} \frac{n+1}{n+2} \Rightarrow \hat{p}_n$ converge to 0 linearly

$\lim_{n\to\infty} \frac{|\hat{a}_n - 0|}{|a_n - 0|} = \frac{1}{2}$ ($\hat{a}_n$ faster than a_n) #

(ii) $b_n = \frac{1}{n^2} \Rightarrow \hat{b}_n = \frac{1}{n^2} - \frac{\left(\frac{1}{(n+1)^2} - \frac{1}{n^2}\right)^2}{\frac{1}{(n+2)^2} - \frac{2}{(n+1)^2} + \frac{1}{n^2}}$

$$= \frac{1}{n^2} - \frac{\left(\frac{-2n-1}{n^2(n+1)^2}\right)^2 (n+2)^2(n+1)^2 n^2}{n^2(n+1)^2 - 2n^2(n+2)^2 + (n+1)^2(n+2)^2}$$

$$= \frac{1}{n^2} - \frac{\frac{(2n+1)^2(n+2)^2}{n^2(n+1)^2}}{6n^2 + 12n + 4}$$

$$= \frac{2n^2 + 4n + 1}{2(n+1)^2(3n^2 + 6n + 2)}$$

clearly, b_n converges to 0 linearly.

and $\lim_{n\to\infty} \frac{|\hat{b}_{n+1}-0|}{|\hat{b}_n-0|} = 1 \Rightarrow \hat{b}_n$ converges to 0 linearly.

$\lim_{n\to\infty} \frac{|\hat{b}_n-0|}{|b_n-0|} = \frac{1}{3}$ ($\hat{b}_n$ faster than b_n) #

(iii) $c_n = \alpha^n \Rightarrow \hat{c}_n = \alpha^n - \frac{(\alpha^{n+1}-\alpha^n)^2}{\alpha^{n+2}-2\alpha^{n+1}+\alpha^n} = \frac{\alpha^{2n+2}-2\alpha^{2n+1}+\alpha^{2n}-\alpha^{2n+2}+2\alpha^{2n+1}-\alpha^{2n}}{\alpha^{n+2}-2\alpha^{n+1}+\alpha^n}$

$(0<\alpha<1)$

$= 0$

$\lim_{n\to\infty} \frac{|c_{n+1}-0|}{|c_n-0|} = \lim_{n\to\infty} \alpha = \alpha \neq 0, \Rightarrow c_n$ converges to 0 linearly.

$\because \hat{c}_n = 0 \quad \therefore \hat{c}_n$ is much more rapidly than c_n #

(iv) $d_n = 2^{-2^n} \Rightarrow \hat{d}_n = 2^{-2^n} - \frac{(2^{-2^{n+1}}-2^{-2^n})^2}{2^{-2^{n+2}}-2\cdot 2^{-2^{n+1}}+2^{-2^n}}$

$= \frac{\left(2^{-2^{n+2}-2^n} - 2\cdot 2^{-2^{n+1}-2^n} + 2^{-2^n-2^n}\right) - \left(2^{-2^{n+1}-2^{n+1}} - 2\cdot 2^{-2^{n+1}-2^n} + 2^{-2^n-2^n}\right)}{2^{-2^{n+2}} - 2\cdot 2^{-2^{n+1}} + 2^{-2^n}}$

$= \frac{2^{-2^{n+2}-2^n} - 2^{-2^{n+1}-2^{n+1}}}{2^{-2^{n+2}}-2\cdot 2^{-2^{n+1}}+2^{-2^n}} = \frac{2^{-2^n(4+1)} - 2^{-2^n(4)}}{2^{-2^n(4)} - 2\cdot 2^{-2^n\cdot 2} + 2^{-2^n}}$

$= \frac{(2^{-2^n})^4 - (2^{-2^n})^3}{(2^{-2^n})^3 - 2(2^{-2^n}) + 1} = \frac{(2^{-2^n})^3}{(2^{-2^n})^2 + (2^{-2^n}) - 1}$

$$\lim_{n\to\infty} \frac{|\alpha_{n+1}-0|}{|\alpha_n-0|^2} = \lim_{n\to\infty} \frac{2^{-2^{n+1}}}{(2^{-2^n})^2} = 1 \Rightarrow \alpha_n \text{ converges to } 0$$

quadratically.

$$\lim_{n\to\infty} \frac{|\hat{\alpha}_{n+1}-0|}{|\hat{\alpha}_n-0|^2} = \lim_{n\to\infty} \left| \frac{\left(2^{-2^{n+1}}\right)^3}{\left(2^{-2^{n+1}}\right)^2+\left(2^{-2^{n+1}}\right)-1} \frac{\left[\left(2^{-2^n}\right)^2+\left(2^{-2^n}\right)-1\right]^2}{\left(2^{-2^n}\right)^6} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\left(2^{-2^n}\right)^6}{\left(2^{-2^n}\right)^6} \cdot \frac{\left[\left(2^{-2^n}\right)^2+\left(2^{-2^n}\right)-1\right]^2}{\left(2^{-2^{n+1}}\right)^2+\left(2^{-2^{n+1}}\right)-1} \right|$$

$= 1 \Rightarrow \hat{\alpha}_n$ converges to 0 quadratically

$$\lim_{n\to\infty} \frac{|\hat{\alpha}_n-0|}{|\alpha_n-0|} = 0$$ ($\hat{\alpha}_n$ is much more rapidly than α_n).

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