HW14, 2

Discrete case : Find $p \in \mathcal{P}_n$ that minimizes $\sum_{j=0}^{m} (p(x_j) - f(x_j))^2$

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

$$\min_{a_0, a_1, \cdots, a_n} \sum_{j=0}^{m} (p(x_j) - f(x_j))^2 \quad , m > n$$

if, $\frac{\partial}{\partial a_i} \sum_{j=0}^{m} (p(x_j) - f(x_j))^2 = \sum_{j=0}^{m} 2(p(x_j) - f(x_j)) x_j^i = 0$

$$\Rightarrow \sum_{j=0}^{m} p(x_j) x_j^i = \sum_{j=0}^{m} f(x_j) x_j^i \quad , \text{and}$$

$$\sum_{j=0}^{m} p(x_j) x_j^i = \left(\begin{array}{l} x_0^i (a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_n x_0^n) \\ + x_1^i (a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_n x_1^n) \\ + \\ \vdots \\ + x_m^i (a_0 + a_1 x_m + a_2 x_m^2 + \cdots + a_n x_m^n) \end{array} \right)$$

$$= \sum_{k=0}^{n} \left[a_k \left(\sum_{j=0}^{m} x_j^{i+k} \right) \right] \quad , \text{and Let } \vec{a} = (a_0, a_1, \cdots, a_n)^t$$

$$\Rightarrow \left(\sum_{j=0}^{m} x_j^i , \sum_{j=0}^{m} x_j^{i+1} , \cdots\cdots , \sum_{j=0}^{m} x_j^{i+n} \right) \vec{a} = \sum_{i=0}^{m} f(x_j) x_j^i$$

for $i = 0, 1, \cdots, n$

$$\Rightarrow \begin{bmatrix} \sum_{j=0}^{m} x_j^0 & \sum_{j=0}^{m} x_j^1 & \sum_{j=0}^{m} x_j^2 & \cdots & \sum_{j=0}^{m} x_j^n \\ \sum_{j=0}^{m} x_j^1 & \sum_{j=0}^{m} x_j^2 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \sum_{j=0}^{m} x_j^n & \cdots & \cdots & \cdots & \sum_{j=0}^{m} x_j^{2n} \end{bmatrix} \vec{a} = \begin{bmatrix} \sum_{i=0}^{m} f(x_j) x_j^0 \\ \sum_{i=0}^{m} f(x_j) x_j^1 \\ \vdots \\ \sum_{i=0}^{m} f(x_j) x_j^n \end{bmatrix}$$

4. (i) Let $f(x) = \alpha x + (1-\alpha)\, g(x)$

if $\exists\ x^* \ni\ x^* = f(x^*) = \alpha x^* + (1-\alpha)\, g(x^*)$

$\Rightarrow (1-\alpha) x^* = (1-\alpha)\, g(x^*)$

if $(1-\alpha) \neq 0$
$\Rightarrow x^* = g(x^*)$

(ii) $g(x) = 1 - 2x + 0.2 \sin x$

$g'(x) = -2 + 0.2 \cos x$

$\Rightarrow -2.2 < g'(x) < -1.8$

$\Rightarrow |g'(x)| > 1$

Now, let $f(x) = \alpha x + (1-\alpha)\, g(x)$

$f'(x) = \alpha + (1-\alpha)\, g'(x)$

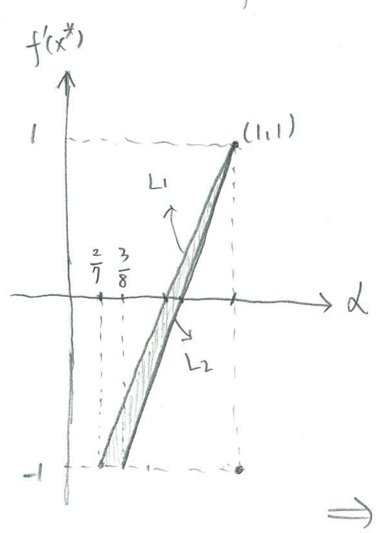
if the fix point exists, say x^*

then $f'(x^*) = \alpha + (1-\alpha)\, g'(x^*)$

we went to choose $\alpha \ni |f'(x^*)| < 1$

Because $-2.2 < g'(x) < -1.8$

So $f'(x^*)$ is between $\underbrace{\alpha + (1-\alpha)(-2.2)}_{\shortparallel\ L_2}$ and $\underbrace{\alpha + (1-\alpha)(-1.8)}_{\shortparallel\ L_1}$



and we may find α minimizes

$$\min_{\alpha} \left(\max \{ |L_1(\alpha)|, |L_2(\alpha)| \} \right)$$

i.e. find $\alpha \ni L_1(\alpha) = -L_2(\alpha)$

$\Rightarrow \alpha = 2/3$ #