

# Exercise Set 7.4, page 461

1. The  $\|\cdot\|_\infty$  condition number is:

- (a) 50                      (b) 241.37                      (c) 600,002                      (d) 339,866

2. The  $\|\cdot\|_\infty$  condition numbers are:

- (a) 12.24012756                      (b) 12.24012756                      (c) 12                      (d) 198.17

3. We have

	$\ \mathbf{x} - \hat{\mathbf{x}}\ _\infty$	$K_\infty(A)\ \mathbf{b} - A\hat{\mathbf{x}}\ _\infty/\ A\ _\infty$
(a)	$8.571429 \times 10^{-4}$	$1.238095 \times 10^{-2}$
(b)	0.1	3.832060
(c)	0.04	0.8
(d)	20	$1.152440 \times 10^5$

4. We have

	$\ \mathbf{x} - \hat{\mathbf{x}}\ _\infty$	$K_\infty(A)\ \mathbf{b} - A\hat{\mathbf{x}}\ _\infty/\ A\ _\infty$
(a)	20	65.03241
(b)	0.02	720.5764
(c)	0.1	$3.727412 \times 10^{-1}$
(d)	$6.551700 \times 10^{-2}$	9.059201

5. Gaussian elimination and iterative refinement give the following results.

- (a) (i)  $(-10.0, 1.01)^t$ , (ii)  $(10.0, 1.00)^t$   
 (b) (i)  $(12.0, 0.499, -1.98)^t$ , (ii)  $(1.00, 0.500, -1.00)^t$   
 (c) (i)  $(0.185, 0.0103, -0.0200, -1.12)^t$ , (ii)  $(0.177, 0.0127, -0.0207, -1.18)^t$   
 (d) (i)  $(0.799, -3.12, 0.151, 4.56)^t$ , (ii)  $(0.758, -3.00, 0.159, 4.30)^t$

6. Gaussian elimination and iterative refinement give the following results.

- (a)  $(12.00, 0.9990)^t$ ,  $(10.00, 1.000)^t$   
 (b)  $(1.200, 0.5002, -1.380)^t$ ,  $(1.000, 0.5000, -0.9998)^t$   
 (c)  $(0.1756, 0.01305, -0.02075, -1.192)^t$ ,  $(0.1768, 0.01269, -0.02065, -1.182)^t$   
 (d)  $(0.7963, -3.152, 0.1705, 4.615)^t$ ,  $(0.7889, -3.128, 0.1678, 4.561)^t$

7. The matrix is ill-conditioned since  $K_\infty = 60002$ . We have  $\tilde{\mathbf{x}} = (-1.0000, 2.0000)^t$ .

8. The matrix  $A$  is ill-conditioned since  $K_\infty(A) = 600,002$  and  $\hat{\mathbf{x}} = (1.818192, 0.5909091)^t$

9. For any vector  $\mathbf{x}$ , we have

$$\|\mathbf{x}\| = \|A^{-1}A\mathbf{x}\| \leq \|A^{-1}\| \|A\mathbf{x}\|, \text{ so } \|A\mathbf{x}\| \geq \frac{\|\mathbf{x}\|}{\|A^{-1}\|}.$$

Let  $\mathbf{x} \neq \mathbf{0}$  be such that  $\|\mathbf{x}\| = 1$  and  $B\mathbf{x} = \mathbf{0}$ . Then

$$\|(A - B)\mathbf{x}\| = \|A\mathbf{x}\| \geq \frac{\|\mathbf{x}\|}{\|A^{-1}\|} \text{ and } \frac{\|(A - B)\mathbf{x}\|}{\|A\|} \geq \frac{1}{\|A^{-1}\| \|A\|} = \frac{1}{K(A)}.$$

Since  $\|\mathbf{x}\| = 1$ ,

$$\|(A - B)\mathbf{x}\| \leq \|A - B\| \|\mathbf{x}\| = \|A - B\| \text{ and } \frac{\|A - B\|}{\|A\|} \geq \frac{1}{K(A)}.$$

10. The approximate condition numbers are as follows:

(a) With  $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ , we have  $K_{\infty}(A) \geq 30,000$ .

(b) With  $B = \begin{bmatrix} 4.0 & 1.6 \\ 7.0 & 2.8 \end{bmatrix}$ , we have  $K_{\infty}(A) \geq \frac{97}{3}$ .

11. (a)  $K_{\infty}(H^{(4)}) = 28,375$

(b)  $K_{\infty}(H^{(5)}) = 943,656$

(c) actual solution  $\mathbf{x} = (-124, 1560, -3960, 2660)^t$ ;  
approximate solution  $\tilde{\mathbf{x}} = (-124.2, 1563.8, -3971.8, 2668.8)^t$ ;  $\|\mathbf{x} - \tilde{\mathbf{x}}\|_{\infty} = 11.8$ ;  $\frac{\|\mathbf{x} - \tilde{\mathbf{x}}\|_{\infty}}{\|\mathbf{x}\|_{\infty}} = 0.02980$ ;

$$\begin{aligned} \frac{K_{\infty}(A)}{1 - K_{\infty}(A) \left( \frac{\|\delta A\|_{\infty}}{\|A\|_{\infty}} \right)} \left[ \frac{\|\delta b\|_{\infty}}{\|b\|_{\infty}} + \frac{\|\delta A\|_{\infty}}{\|A\|_{\infty}} \right] &= \frac{28375}{1 - 28375 \left( \frac{6.6 \times 10^{-6}}{2.083} \right)} \left[ 0 + \frac{6.6 \times 10^{-6}}{2.083} \right] \\ &= 0.09987. \end{aligned}$$

12. For the  $3 \times 3$  Hilbert matrix  $H$ , we have

$$\hat{H}^{-1} = \begin{bmatrix} 8.968 & -35.77 & 29.77 \\ -35.77 & 190.6 & -178.6 \\ 29.77 & -178.6 & 178.6 \end{bmatrix}, \quad \hat{H} = \begin{bmatrix} 0.9799 & 0.4870 & 0.3238 \\ 0.4860 & 0.3246 & 0.2434 \\ 0.3232 & 0.2433 & 0.1949 \end{bmatrix},$$

and  $\|H - \hat{H}\|_{\infty} = 0.04260$ .