Exercise Set 7.3, page 449

1. Two iterations of Jacobi's method gives the following results.

   (a) $\mathbf{x}^{(2)} = (0.1428571, -0.3571429, 0.4285714)^t$ (b) $\mathbf{x}^{(2)} = (0.97, 0.91, 0.74)^t$

   (c) $\mathbf{x}^{(2)} = (-0.65, 1.65, -0.4, -2.475)^t$ (d) $\mathbf{x}^{(2)} = (1.325, -1.6, 1.6, 1.675, 2.425)^t$

2. Two iterations of Jacobi's method gives the following results.
   (a) $\mathbf{x}^{(2)} = (1.2500000, -1.3333333, 0.2000000)^t$
   (b) $\mathbf{x}^{(2)} = (-1.0000000, 1.0000000, -1.3333333)^t$
   (c) $\mathbf{x}^{(2)} = (-0.5208333, -0.04166667, -0.2166667, 0.4166667)^t$
   (d) $\mathbf{x}^{(2)} = (0.6875, 1.125, 0.6875, 1.375, 0.5625, 1.375)^t$

3. Two iterations of the Gauss-Seidel method give the following results.
   (a) $\mathbf{x}^{(2)} = (0.1111111, -0.2222222, 0.6190476)^t$
   (b) $\mathbf{x}^{(2)} = (0.979, 0.9495, 0.7899)^t$
   (c) $\mathbf{x}^{(2)} = (-0.5, 2.64, -0.336875, -2.267375)^t$
   (d) $\mathbf{x}^{(2)} = (1.189063, -1.521354, 1.862396, 1.882526, 2.255645)^t$

4. Two iterations of the Gauss-Seidel method give the following results.
   (a) $\mathbf{x}^{(2)} = (1.250000000, -0.9166666667, 0.06666666666)^t$
   (b) $\mathbf{x}^{(2)} = (-1.666666667, 1.333333334, -0.8888888894)^t$
   (c) $\mathbf{x}^{(2)} = (-0.625, 0, -0.225, 0.6166667)^t$
   (d) $\mathbf{x}^{(2)} = (0.6875, 1.546875, 0.7929688, 1.71875, 0.7226563, 1.878906)^t$

5. Jacobi's Algorithm gives the following results.
   (a) $\mathbf{x}^{(10)} = (0.03507839, -0.2369262, 0.6578015)^t$
   (b) $\mathbf{x}^{(6)} = (0.9957250, 0.9577750, 0.7914500)^t$
   (c) $\mathbf{x}^{(22)} = (-0.7975853, 2.794795, -0.2588888, -2.251879)^t$
   (d) $\mathbf{x}^{(14)} = (-0.7529267, 0.04078538, -0.2806091, 0.6911662)^t$
   (e) $\mathbf{x}^{(12)} = (0.7870883, -1.003036, 1.866048, 1.912449, 1.985707)^t$
   (f) $\mathbf{x}^{(17)} = (0.9996805, 1.999774, 0.9996805, 1.999840, 0.9995482, 1.999840)^t$

6. Jacobi's Algorithm gives the following results.
   (a) $\mathbf{x}^{(10)} = (1.447642384, -0.8355647882, -0.0450226618)^t$
   (b) $\mathbf{x}^{(25)} = (-1.500322611, 1.500322611, -0.9997048580)^t$
   (c) $\mathbf{x}^{(14)} = (-0.7529267, 0.04078538, -0.2806091, 0.6911662)^t$
   (d) $\mathbf{x}^{(17)} = (0.9996805, 1.999774, 0.9996805, 1.999840, 0.9995482, 1.999840)^t$

7. The Gauss-Seidel Algorithm gives the following results.

   (a) $\mathbf{x}^{(6)} = (0.03535107, -0.2367886, 0.6577590)^t$
   (b) $\mathbf{x}^{(4)} = (0.9957475, 0.9578738, 0.7915748)^t$
   (c) $\mathbf{x}^{(10)} = (-0.7973091, 2.794982, -0.2589884, -2.251798)^t$
   (d) $\mathbf{x}^{(7)} = (0.7866825, -1.002719, 1.866283, 1.912562, 1.989790)^t$

8. The Gauss-Seidel Algorithm gives the following results.

   (a) $\mathbf{x}^{(6)} = (1.447816350, -0.8358173037, -0.0447996186)^t$
   (b) $\mathbf{x}^{(8)} = (-1.500228624, 1.499713760, -0.9998475841)^t$
   (c) $\mathbf{x}^{(8)} = (-0.7531763, 0.04101049, -0.2807047, 0.6916305)^t$
   (d) $\mathbf{x}^{(10)} = (0.9998334, 1.999858, 0.9999393, 1.999899, 0.9999142, 1.999963)^t$

9. Two iterations of the SOR method with $\omega = 1.1$ give the following results.

   (a) $\mathbf{x}^{(2)} = (0.05410079, -0.2115435, 0.6477159)^t$
   (b) $\mathbf{x}^{(2)} = (0.9876790, 0.9784935, 0.7899328)^t$
   (c) $\mathbf{x}^{(2)} = (-0.71885, 2.818822, -0.2809726, -2.235422)^t$
   (d) $\mathbf{x}^{(2)} = (1.079675, -1.260654, 2.042489, 1.995373, 2.049536)^t$

10. Two iterations of the SOR method with $\omega = 1.1$ give the following results.

   (a) $\mathbf{x}^{(2)} = (1.512775000, -0.8298491667, -0.0843373667)^t$
   (b) $\mathbf{x}^{(2)} = (-1.58523750, 1.37885688, -0.7039212812)^t$
   (c) $\mathbf{x}^{(2)} = (-0.6604902, 0.03700749, -0.2493513, 0.6561139)^t$
   (d) $\mathbf{x}^{(2)} = (0.3781250000, 1.445468750, 0.3596914062, 1.458531250, 0.3071921875, 1.572124727)^t$

11. Two iterations of the SOR method with $\omega = 1.3$ give the following results.

   (a) $\mathbf{x}^{(2)} = (-0.1040103, -0.1331814, 0.6774997)^t$
   (b) $\mathbf{x}^{(2)} = (0.957073, 0.9903875, 0.7206569)^t$
   (c) $\mathbf{x}^{(2)} = (-1.23695, 3.228752, -0.1523888, -2.041266)^t$
   (d) $\mathbf{x}^{(2)} = (0.7064258, -0.4103876, 2.417063, 2.251955, 1.061507)^t$

12. Two iterations of the SOR method with $\omega = 1.3$ give the following results.

   (a) $\mathbf{x}^{(2)} = (1.455783334, -0.7721494442, -0.0805396228)^t$
   (b) $\mathbf{x}^{(2)} = (-1.42073750, 1.595758125, -0.8597927812)^t$
   (c) $\mathbf{x}^{(2)} = (-0.7268893, 0.1251483, -0.2923371, 0.7037018)^t$
   (d) $\mathbf{x}^{(2)} = (0.5281250000, 1.480781250, 0.322816406, 1.359718750, 0.4288171875, 1.505949961)^t$

13. The SOR Algorithm with $\omega = 1.2$ gives the following results.

   (a) $\mathbf{x}^{(12)} = (0.03488469, -0.2366474, 0.6579013)^t$
   (b) $\mathbf{x}^{(7)} = (0.9958341, 0.9579041, 0.7915756)^t$

(c) $\mathbf{x}^{(8)} = (-0.7976009, 2.795288, -0.2588293, -2.251768)^t$

(d) $\mathbf{x}^{(7)} = (-0.7534489, 0.04106617, -0.2808146, 0.6918049)^t$

(e) $\mathbf{x}^{(10)} = (0.7866310, -1.002807, 1.866530, 1.912645, 1.989792)^t$

(f) $\mathbf{x}^{(7)} = (0.9999442, 1.999934, 1.000033, 1.999958, 0.9999815, 2.000007)^t$

14. The SOR Algorithm with $\omega = 1.2$ gives the following results.

(a) $\mathbf{x}^{(8)} = (1.447503814, -0.8359297624, -0.0445516532)^t$

(b) $\mathbf{x}^{(6)} = (-1.454582850, 1.454498863, -0.7273302714)^t$

(c) $\mathbf{x}^{(7)} = (-0.7534489, 0.04106617, -0.2808146, 0.6918049)^t$

(d) $\mathbf{x}^{(7)} = (0.3571284945, 1.428582240, 0.3571489731, 1.571440116, 0.2857000650, 1.571445036)^t$

15. The tridiagonal matrices are in parts (b) and (c).
(9b): For $\omega = 1.012823$ we have $\mathbf{x}^{(4)} = (0.9957846, 0.9578935, 0.7915788)^t$.
(9c): For $\omega = 1.153499$ we have $\mathbf{x}^{(7)} = (-0.7977651, 2.795343, -0.2588021, -2.251760)^t$.

16. The tridiagonal matrix is in part (d).
(10d): For $\omega = 1.033370453$ we have

$$\mathbf{x}^{(5)} = (0.3571407017, 1.428570817, 0.357142771, 1.571421010, 0.2857118407, 1.571428256)^t.$$

17. (a)

$$T_j = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \quad \text{and} \quad \det(\lambda I - T_j) = \lambda^3 + \frac{5}{4}x.$$

Thus, the eigenvalues of T_j are 0 and $\pm\frac{\sqrt{5}}{2}i$, so $\rho(T_j) = \frac{\sqrt{5}}{2} > 1$.

(b) $\mathbf{x}^{(25)} = (-20.827873, 2.0000000, -22.827873)^t$

(c)

$$T_g = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \quad \text{and} \quad \det(\lambda I - T_g) = \lambda\left(\lambda + \frac{1}{2}\right)^2.$$

Thus, the eigenvalues of T_g are 0, $-\frac{1}{2}$, and $-\frac{1}{2}$; and $\rho(T_g) = \frac{1}{2}$.

(d) $\mathbf{x}^{(23)} = (1.0000023, 1.9999975, -1.0000001)^t$ is within 10^{-5} in the l_∞ norm.

18. (a) $T_j = \begin{bmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix}$ and $\det(\lambda I - T_j) = \lambda^3$, so $\rho(T_j) = 0$.

(b) $\mathbf{x}^{(4)} = (1.00000000, 2.00000000, -1.00000000)^t$ is within 10^{-5} in the l_∞ norm.

(c) $T_g = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{bmatrix}$ and $\det(\lambda I - T_g) = \lambda(\lambda - 2)^2$, so $\rho(T_g) = 2$.

(d) $\mathbf{x}^{(25)} = \left(1.30 \times 10^9, -1.325 \times 10^9, 3.355 \times 10^7\right)^t$

19. (a) A is not strictly diagonally dominant.

(b)

$$T_j = \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 0 & 0.25 \\ -1 & 0.5 & 0 \end{bmatrix} \quad \text{and} \quad \rho(T_j) = 0.97210521.$$

Since T_j is convergent, the Jacobi method will converge.

(c) With $\mathbf{x}^{(0)} = (0,0,0)^t$, $\mathbf{x}^{(187)} = (0.90222655, -0.79595242, 0.69281316)^t$

(d) $\rho(T_j) = 1.39331779371$. Since T_j is not convergent, the Jacobi method will not converge.

20. (a) A is not strictly diagonally dominant.

(b) We have

$$T_j = \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 0 & 0.25 \\ -1 & 0.5 & 0 \end{bmatrix} \quad \text{and} \quad \rho(T_j) = 0.97210521.$$

Since T_j is convergent, the Jacobi method will converge.

(c) With $\mathbf{x}^{(0)} = (0,0,0)^t$, $\mathbf{x}^{(187)} = (0.90222655, -0.79595242, 0.69281316)^t$

(d) $\rho(T_j) = 1.39331779371$. Since T_j is not convergent, the Jacobi method will not converge.

21. (a) Subtract $\mathbf{x} = T\mathbf{x} + \mathbf{c}$ from $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$ to obtain $\mathbf{x}^{(k)} - \mathbf{x} = T(\mathbf{x}^{(k-1)} - \mathbf{x})$. Thus,

$$\|\mathbf{x}^{(k)} - \mathbf{x}\| \le \|T\| \, \|\mathbf{x}^{(k-1)} - \mathbf{x}\|.$$

Inductively, we have

$$\|\mathbf{x}^{(k)} - \mathbf{x}\| \le \|T\|^k \|\mathbf{x}^{(0)} - \mathbf{x}\|.$$

The remainder of the proof is similar to the proof of Corollary 2.5.

(b) The last column has no entry when $\|T\|_\infty = 1$.

	$\|\mathbf{x}^{(2)} - \mathbf{x}\|_\infty$	$\|T\|_\infty$	$\|T\|_\infty^2 \|\mathbf{x}^{(0)} - \mathbf{x}\|_\infty$	$\frac{\|T\|_\infty^2}{1-\|T\|_\infty} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|_\infty$
1 (a)	0.22932	0.857143	0.48335	2.9388
1 (b)	0.051579	0.3	0.089621	0.11571
1 (c)	1.1453	0.9	2.2642	20.25
1 (d)	0.27511	1	0.75342	
1 (e)	0.59743	1	1.9897	
1 (f)	0.875	0.75	1.125	3.375

22. The matrix $T_j = (t_{ik})$ has entries given by

$$t_{ik} = \begin{cases} 0, & i = k \text{ for } 1 \le i \le n \text{ and } 1 \le k \le n \\ -\frac{a_{ik}}{a_{ii}}, & i \ne k \text{ for } 1 \le i \le n \text{ and } 1 \le k \le n. \end{cases}$$

Since A is strictly diagonally dominant,

$$\|T_j\|_\infty = \max_{1 \le i \le n} \sum_{\substack{k=1 \\ k \ne i}}^{n} \left| \frac{a_{ik}}{a_{ii}} \right| < 1.$$

23. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of T_ω. Then

$$\prod_{i=1}^{n} \lambda_i = \det T_\omega = \det\left((D-\omega L)^{-1}[(1-\omega)D+\omega U]\right)$$
$$= \det(D-\omega L)^{-1}\det((1-\omega)D+\omega U) = \det\left(D^{-1}\right)\det((1-\omega)D)$$
$$= \left(\frac{1}{(a_{11}a_{22}\ldots a_{nn})}\right)\left((1-\omega)^n a_{11}a_{22}\ldots a_{nn})\right) = (1-\omega)^n.$$

Thus,

$$\rho(T_\omega) = \max_{1\le i\le n} |\lambda_i| \ge |\omega - 1|,$$

and $|\omega - 1| < 1$ if and only if $0 < \omega < 2$.

24. (a) We have $P_0 = 1$, so the equation $P_1 = \frac{1}{2}P_0 + \frac{1}{2}P_2$ gives $P_1 - \frac{1}{2}P_2 = \frac{1}{2}$. Since $P_i = \frac{1}{2}P_{i-1} + \frac{1}{2}P_{i+1}$, we have $-\frac{1}{2}P_{i-1} + P_i - \frac{1}{2}P_{i+1} = 0$, for $i = 2, \ldots, n-2$. Finally, since $P_n = 0$ and $P_{n-1} = \frac{1}{2}P_{n-2} + \frac{1}{2}P_n$, we have $-\frac{1}{2}P_{n-2} + P_{n-1} = 0$. This gives the linear system.

(b) The solution vector is $(0.89996431, 0.79993544, 0.69991549, 0.59990552, 0.49990552, 0.39991454, 0.29993086, 0.19995223, 0.09997611)^t$, using 86 iterations with a tolerance 1.00×10^{-5} in l_∞ with the Gauss-Seidel method.

The solution vector is $(0.96289774, 0.92595527, 0.88925042, 0.85285897, 0.81685427, 0.78130672, 0.74628346, 0.71184798, 0.67805979, 0.64497421, 0.61264206, 0.58110953, 0.55041801, 0.52060401, 0.49169906, 0.46372973, 0.43671763, 0.41067944, 0.38562707, 0.36156768, 0.33850391, 0.31643400, 0.29535198, 0.27524791, 0.25610805, 0.23791514, 0.22064859, 0.20428475, 0.18879715, 0.17415669, 0.16033195, 0.14728936, 0.13499341, 0.12340690, 0.11249111, 0.10220596, 0.09251023, 0.08336165, 0.07471709, 0.06653267, 0.05876386, 0.05136562, 0.04429243, 0.03749843, 0.03093747, 0.02456315, 0.01832893, 0.01218814, 0.00609407)^t$, using 231 iterations with tolerance 1.00×10^{-3} in l_∞ with the Gauss-Seidel method.

The solution vector is $(0.96305854, 0.92627494, 0.88972613, 0.85348706, 0.81763026, 0.78222543, 0.74733909, 0.71303418, 0.67936983, 0.64640101, 0.61417841, 0.58274816, 0.55215178, 0.52242602, 0.49360287, 0.46570950, 0.43876832, 0.41279701, 0.38780868, 0.36381196, 0.34081114, 0.31880642, 0.29779408, 0.27776668, 0.25871338, 0.24062014, 0.22346997, 0.20724328, 0.19191807, 0.17747025, 0.16387393, 0.15110162, 0.13912457, 0.12791297, 0.11743622, 0.10766312, 0.09856216, 0.09010163, 0.08224988, 0.07497547, 0.06824731, 0.06203481, 0.05630801, 0.05103770, 0.04619548, 0.04175387, 0.03768638, 0.03396754, 0.03057293, 0.02747926, 0.02466435, 0.02210715, 0.01978772, 0.01768725, 0.01578806, 0.01407350, 0.01252803, 0.01113710, 0.00988718, 0.00876568, 0.00776092, 0.00686210, 0.00605926, 0.00534321, 0.00470552, 0.00413844, 0.00363490, 0.00318842, 0.00279312, 0.00244363, 0.00213509, 0.00186308, 0.00162362, 0.00141311, 0.00122831, 0.00106630, 0.00092447, 0.00080047, 0.00069221, 0.00059781, 0.00051560, 0.00044409, 0.00038197, 0.00032806, 0.00028132, 0.00024082, 0.00020575, 0.00017539, 0.00014909, 0.00012629, 0.00010648, 0.00008920, 0.00007405, 0.00006067, 0.00004871, 0.00003787, 0.00002786, 0.00001839, 0.00000919)^t$, using 233 iterations with tolerance 1.00×10^{-3} in l_∞ norm with the Gauss-Seidel method.

(c) The equations are $P_i = \alpha P_{i-1} + (1-\alpha)P_{i+1}$, for $i = 1, 2, \ldots, n-1$, and the linear system becomes

$$\begin{bmatrix} 1 & \alpha-1 & 0 & \cdots & 0 \\ -\alpha & 1 & \alpha-1 & \ddots & \vdots \\ 0 & -\alpha & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \alpha-1 \\ 0 & \cdots & 0 & -\alpha & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ \vdots \\ P_{n-1} \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

(d) The solution vector is $(0.49947985, 0.24922901, 0.12411164, 0.06155895, 0.03028662, 0.01465286, 0.00683728, 0.00293009, 0.00097670)^t$,
using 35 iterations with tolerance 1.00×10^{-5} in l_∞ norm with the Gauss-Seidel method.
The solution vector is $(4.9995328 \times 10^{-1}, 2.4993967 \times 10^{-1}, 1.2494215 \times 10^{-1}, 6.2451172\times10^{-2}, 3.1211719\times10^{-2}, 1.5596448\times10^{-2}, 7.7919757\times10^{-3}, 3.8919193\times10^{-3}, 1.9433556\times10^{-3}, 9.7003673\times10^{-4}, 4.8399876\times10^{-4}, 2.4137415\times10^{-4}, 1.2030832\times10^{-4}, 5.9927321\times10^{-5}, 2.9829260\times10^{-5}, 1.4835803\times10^{-5}, 7.3721140\times10^{-6}, 3.6597037\times10^{-6}, 1.8148167\times10^{-6}, 8.9890229\times10^{-7}, 4.4467752\times10^{-7}, 2.1968048\times10^{-7}, 1.0837093\times10^{-7}, 5.3379165\times10^{-8}, 2.6250177\times10^{-8}, 1.2887232\times10^{-8}, 6.3156953\times10^{-9}, 3.0894829\times10^{-9}, 1.5084277\times10^{-9}, 7.3503876\times10^{-10}, 3.5745232\times10^{-10}, 1.7347035\times10^{-10}, 8.4006105\times10^{-11}, 4.0593470\times10^{-11}, 1.9572418\times10^{-11}, 9.4158798\times10^{-12}, 4.5195197\times10^{-12}, 2.1643465\times10^{-12}, 1.0340770\times10^{-12}, 4.9289757\times10^{-13}, 2.3437677\times10^{-13}, 1.1116644\times10^{-13}, 5.2577488\times10^{-14}, 2.4776509\times10^{-14}, 1.1608190\times10^{-14}, 5.3767458\times10^{-15}, 2.4249977\times10^{-15}, 1.0192489\times10^{-15}, 3.3974965 \times 10^{-16})^t$,
using 40 iterations with tolerance 1.00×10^{-5} in l_∞ norm with the Gauss-Seidel method.
The solution vector is $(4.9995328 \times 10^{-1}, 2.4993967 \times 10^{-1}, 1.2494215 \times 10^{-1}, 6.2451172\times10^{-2}, 3.1211719\times10^{-2}, 1.5596448\times10^{-2}, 7.7919757\times10^{-3}, 3.8919193\times10^{-3}, 1.9433556\times10^{-3}, 9.7003673\times10^{-4}, 4.8399876\times10^{-4}, 2.4137415\times10^{-4}, 1.2030832\times10^{-4}, 5.9927321\times10^{-5}, 2.9829260\times10^{-5}, 1.4835803\times10^{-5}, 7.3721140\times10^{-6}, 3.6597037\times10^{-6}, 1.8148167\times10^{-6}, 8.9890229\times10^{-7}, 4.4467752\times10^{-7}, 2.1968048\times10^{-7}, 1.0837093\times10^{-7}, 5.3379165\times10^{-8}, 2.6250177\times10^{-8}, 1.2887232\times10^{-8}, 6.3156953\times10^{-9}, 3.0894829\times10^{-9}, 1.5084277\times10^{-9}, 7.3503876\times10^{-10}, 3.5745232\times10^{-10}, 1.7347035\times10^{-10}, 8.4006106\times10^{-11}, 4.0593472\times10^{-11}, 1.9572421\times10^{-11}, 9.4158848\times10^{-12}, 4.5195275\times10^{-12}, 2.1643581\times10^{-12}, 1.0340940\times10^{-12}, 4.9292167\times10^{-13}, 2.3441025\times10^{-13}, 1.1121189\times10^{-13}, 5.2637877\times10^{-14}, 2.4855116\times10^{-14}, 1.1708532\times10^{-14}, 5.5024789\times10^{-15}, 2.5797915\times10^{-15}, 1.2066549\times10^{-15}, 5.6306241\times10^{-16}, 2.6212505\times10^{-16}, 1.2174281\times10^{-16}, 5.6411249\times10^{-17}, 2.6078415\times10^{-17}, 1.2028063\times10^{-17}, 5.5349743\times10^{-18}, 2.5412522\times10^{-18}, 1.1641243\times10^{-18}, 5.3208120\times10^{-19}, 2.4265609\times10^{-19}, 1.1041988\times10^{-19}, 5.0136548\times10^{-20}, 2.2715436\times10^{-20}, 1.0269655\times10^{-20}, 4.6330552\times10^{-21}, 2.0857623\times10^{-21}, 9.3703715\times10^{-22}, 4.2009978\times10^{-22}, 1.8795800\times10^{-22}, 8.3924933\times10^{-23}, 3.7398320\times10^{-23}, 1.6632335\times10^{-23}, 7.3825009\times10^{-24}, 3.2704840\times10^{-24}, 1.4460652\times10^{-24}, 6.3817537\times10^{-25}, 2.8111081\times10^{-25}, 1.2359739\times10^{-25}, 5.4243064\times10^{-26}, 2.3762443\times10^{-26}, 1.0391031\times10^{-26}, 4.5358179\times10^{-27}, 1.9764714\times10^{-27}, 8.5974956\times10^{-28}, 3.7334326\times10^{-28}, 1.6184823\times10^{-28}, 7.0045319\times10^{-29}, 3.0264255\times10^{-29}, 1.3054753\times10^{-29}, 5.6221577\times10^{-30}, 2.4173573\times10^{-30}, 1.0377414\times10^{-30}, 4.4478726\times10^{-31}, 1.9033751\times10^{-31}, 8.1312453\times10^{-32}, 3.4660513\times10^{-32}, 1.4712665\times10^{-32}, 6.1707325\times10^{-33}, 2.4790812\times10^{-33}, 8.2636039 \times 10^{-34})^t$,
using 40 iterations with tolerance 1.00×10^{-5} in l_∞ norm with the Gauss-Seidel method.

25.

	Jacobi 33 iterations	Gauss-Seidel 8 iterations	SOR ($\omega = 1.2$) 13 iterations
x_1	1.53873501	1.53873270	1.53873549
x_2	0.73142167	0.73141966	0.73142226
x_3	0.10797136	0.10796931	0.10797063
x_4	0.17328530	0.17328340	0.17328480
x_5	0.04055865	0.04055595	0.04055737
x_6	0.08525019	0.08524787	0.08524925
x_7	0.16645040	0.16644711	0.16644868
x_8	0.12198156	0.12197878	0.12198026
x_9	0.10125265	0.10124911	0.10125043
x_{10}	0.09045966	0.09045662	0.09045793
x_{11}	0.07203172	0.07202785	0.07202912
x_{12}	0.07026597	0.07026266	0.07026392
x_{13}	0.06875835	0.06875421	0.06875546
x_{14}	0.06324659	0.06324307	0.06324429
x_{15}	0.05971510	0.05971083	0.05971200
x_{16}	0.05571199	0.05570834	0.05570949
x_{17}	0.05187851	0.05187416	0.05187529
x_{18}	0.04924911	0.04924537	0.04924648
x_{19}	0.04678213	0.04677776	0.04677885
x_{20}	0.04448679	0.04448303	0.04448409
x_{21}	0.04246924	0.04246493	0.04246597
x_{22}	0.04053818	0.04053444	0.04053546
x_{23}	0.03877273	0.03876852	0.03876952
x_{24}	0.03718190	0.03717822	0.03717920
x_{25}	0.03570858	0.03570451	0.03570548
x_{26}	0.03435107	0.03434748	0.03434844
x_{27}	0.03309542	0.03309152	0.03309246
x_{28}	0.03192212	0.03191866	0.03191958
x_{29}	0.03083007	0.03082637	0.03082727
x_{30}	0.02980997	0.02980666	0.02980755
x_{31}	0.02885510	0.02885160	0.02885248
x_{32}	0.02795937	0.02795621	0.02795707
x_{33}	0.02711787	0.02711458	0.02711543
x_{34}	0.02632478	0.02632179	0.02632262

	Jacobi 33 iterations	Gauss-Seidel 8 iterations	SOR ($\omega = 1.2$) 13 iterations
x_{35}	0.02557705	0.02557397	0.02557479
x_{36}	0.02487017	0.02486733	0.02486814
x_{37}	0.02420147	0.02419858	0.02419938
x_{38}	0.02356750	0.02356482	0.02356560
x_{39}	0.02296603	0.02296333	0.02296410
x_{40}	0.02239424	0.02239171	0.02239247
x_{41}	0.02185033	0.02184781	0.02184855
x_{42}	0.02133203	0.02132965	0.02133038
x_{43}	0.02083782	0.02083545	0.02083615
x_{44}	0.02036585	0.02036360	0.02036429
x_{45}	0.01991483	0.01991261	0.01991324
x_{46}	0.01948325	0.01948113	0.01948175
x_{47}	0.01907002	0.01906793	0.01906846
x_{48}	0.01867387	0.01867187	0.01867239
x_{49}	0.01829386	0.01829190	0.01829233
x_{50}	0.71792896	0.01792707	0.01792749
x_{51}	0.01757833	0.01757648	0.01757683
x_{52}	0.01724113	0.01723933	0.01723968
x_{53}	0.01691660	0.01691487	0.01691517
x_{54}	0.01660406	0.01660237	0.01660267
x_{55}	0.01630279	0.01630127	0.01630146
x_{56}	0.01601230	0.01601082	0.01601101
x_{57}	0.01573198	0.01573087	0.01573077
x_{58}	0.01546129	0.01546020	0.01546010
x_{59}	0.01519990	0.01519909	0.01519878
x_{60}	0.01494704	0.01494626	0.01494595
x_{61}	0.01470181	0.01470085	0.01470077
x_{62}	0.01446510	0.01446417	0.01446409
x_{63}	0.01423556	0.01423437	0.01423461
x_{64}	0.01401350	0.01401233	0.01401256
x_{65}	0.01380328	0.01380234	0.01380242
x_{66}	0.01359448	0.01359356	0.01359363
x_{67}	0.01338495	0.01338434	0.01338418
x_{68}	0.01318840	0.01318780	0.01318765
x_{69}	0.01297174	0.01297109	0.01297107
x_{70}	0.01278663	0.01278598	0.01278597
x_{71}	0.01270328	0.01270263	0.01270271
x_{72}	0.01252719	0.01252656	0.01252663
x_{73}	0.01237700	0.01237656	0.01237654
x_{74}	0.01221009	0.01220965	0.01220963
x_{75}	0.01129043	0.01129009	0.01129008
x_{76}	0.01114138	0.01114104	0.01114102
x_{77}	0.01217337	0.01217312	0.01217313
x_{78}	0.01201771	0.01201746	0.01201746
x_{79}	0.01542910	0.01542896	0.01542896
x_{80}	0.01523810	0.01523796	0.01523796

26. (a) Since A is a positive definite, $a_{ii} > 0$ for $1 \le i \le n$ and A is symmetric. Thus, A can be written as $A = D - L - L^t$, where D is diagonal with $d_{ii} > 0$ and L is lower triangular. The diagonal of the lower triangular matrix $D - L$ has the positive entries $d_{11} = a_{11}$, $d_{22} = a_{22}, \cdots, d_{nn} = a_{nn}$, so $(D - L)^{-1}$ exists.

(b) Since A is symmetric,

$$P^t = \left(A - T_g^t A T_g\right)^t = A^t - T_g^t A^t T_g = A - T_g^t A T_g = P.$$

Thus, P is symmetric.

(c) $T_g = (D - L)^{-1} L^t$, so

$$(D - L)T_g = L^t = D - L - D + L + L^t = (D - L) - (D - L - L^t) = (D - L) - A.$$

Since $(D - L)^{-1}$ exists, we have $T_g = I - (D - L)^{-1} A$.

(d) Since $Q = (D - L)^{-1} A$, we have $T_g = I - Q$. Note that Q^{-1} exists. By the definition of P we have

$$\begin{aligned} P =& A - T_g^t A T_g = A - \left[I - (D - L)^{-1} A\right]^t A \left[I - (D - L)^{-1} A\right] \\ =& A - [I - Q]^t A [I - Q] = A - \left(I - Q^t\right) A (I - Q) \\ =& A - \left(A - Q^t A\right)(I - Q) = A - \left(A - Q^t A - AQ + Q^t AQ\right) \\ =& Q^t A + AQ - Q^t AQ = Q^t \left[A + \left(Q^t\right)^{-1} AQ - AQ\right] \\ =& Q^t \left[AQ^{-1} + \left(Q^t\right)^{-1} A - A\right] Q. \end{aligned}$$

(e) Since

$$AQ^{-1} = A\left[A^{-1}(D - L)\right] = D - L$$

and

$$\left(Q^t\right)^{-1} A = D - L^t,$$

we have

$$AQ^{-1} + \left(Q^t\right)^{-1} A - A = D - L + D - L^t - \left(D - L - L^t\right) = D.$$

Thus,

$$P = Q^t \left[AQ^{-1} + \left(Q^t\right)^{-1} A - A\right] Q = Q^t D Q.$$

So for $\mathbf{x} \in \mathbb{R}^n$, we have $\mathbf{x}^t P \mathbf{x} = \mathbf{x}^t Q^t D Q \mathbf{x} = (Q\mathbf{x})^t D (Q\mathbf{x})$.
Since D is a positive diagonal matrix, $(Q\mathbf{x})^t D (Q\mathbf{x}) \ge 0$ unless $Q\mathbf{x} = \mathbf{0}$. However, Q is nonsingular, so $Q\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$. Thus, P is positive definite.

(f) Let λ be an eigenvalue of T_g with the eigenvector $\mathbf{x} \ne \mathbf{0}$. Since $\mathbf{x}^t P \mathbf{x} > 0$,

$$\mathbf{x}^t \left[A - T_g^t A T_g\right] \mathbf{x} > 0$$

and

$$\mathbf{x}^t A \mathbf{x} - \mathbf{x}^t T_g^t A T_g \mathbf{x} > 0.$$

Since $T_g\mathbf{x} = \lambda\mathbf{x}$, we have $\mathbf{x}^tT_g^t = \lambda\mathbf{x}^t$ so

$$\left(1-\lambda^2\right)\mathbf{x}^tAx = \mathbf{x}^tA\mathbf{x} - \lambda^2\mathbf{x}^tAx > 0.$$

Since A is positive definite, $1-\lambda^2 > 0$, and $\lambda^2 < 1$. Thus $|\lambda| < 1$.

(g) For any eigenvalue λ of T_g, we have $|\lambda| < 1$. This implies $\rho(T_g) < 1$ and T_g is convergent.

27. For $0 < \omega < 2$, let $T_\omega = (D-\omega L)^{-1}\left[(1-\omega)D + \omega L^t\right]$. Let $P = A - T_\omega^t A T_\omega$ and note that P is symmetric.

As in Exercise 16 we derive a new representation for T_ω:

$$(D-\omega L)T_\omega = (1-\omega)D + \omega L^t = (D-\omega L) - \omega A, \quad \text{so} \quad T_\omega = I - \omega(D-\omega L)^{-1}A.$$

Let $Q = \omega(D-\omega L)^{-1}A$ and $Q^t = \omega A\left[(D-\omega L)^{-1}\right]^t$.
Again $P = Q^t\left[AQ^{-1} + \left(Q^t\right)^{-1}A - A\right]Q$. But $AQ^{-1} = \frac{1}{\omega}(D-\omega L)$ and $\left(Q^t\right)^{-1}A = \frac{1}{\omega}\left(D-\omega L^t\right)$ so that

$$\begin{aligned} AQ^{-1} + \left(Q^t\right)^{-1}A - A &= \frac{1}{\omega}\left[D - \omega L + D - \omega L^t\right] - A \\ &= \frac{2}{\omega}D - D + D - L - L^t - A \\ &= \left(\frac{2}{\omega} - 1\right)D. \end{aligned}$$

Thus $P = \left(\frac{2}{\omega} - 1\right)Q^tDQ$. Since $0 < \omega < 2$, we have $\frac{2}{\omega} - 1 > 0$ and P is positive definite. This proof follows Exercise 16 with T_g replaced by T_ω. Hence, T_ω is convergent.

28. (a) The system was reordered so that the diagonal of the matrix had nonzero entries.

(b) (i) The solution vector using 30 iterations is

$$(0.00362, -6339.744638, -3660.253272, -8965.755808, 6339.744638, 10000, \\ -7320.508959, 6339.746729)^t.$$

(ii) The solution vector using 57 iterations is

$$(-0.002651, -6339.744637, -3660.255362, -8965.752851, 6339.748259, 10000, \\ -7320.506544, 6339.748258)^t.$$

(iii) Method does not converge using $\omega = 1.25$. However, using $\omega = 1.1$ and using 132 iterations gives the solution vector

$$(0.0045175, -6339.744528, -3660.253009, -8965.756179, 6339.743756, 10000, \\ -7320.509547, 6339.747544)^t.$$