HW - week 10

2. LDL^T Factorization

From textbook P404, algorithm 6.5

Step 1 For $i = 1, \cdots, n$ do step 2-4

Step 2 For $j = 1, \cdots, i-1$, set $v_j = l_{ij} d_j$

Step 3 Set $d_i = a_{ii} - \sum_{j=1}^{i-1} l_{ij} v_j$

Step 4. For $j = i+1, \cdots, n$, set $l_{ji} = (a_{ji} - \sum_{k=1}^{i-1} l_{jk} v_k)/d_i$.

i	1	2	⋯	m	⋯	n
step2	0	1	⋯	m-1	⋯	n-1
step3	0	1	⋯	m-1	⋯	n-1
step4	n-1	(n-2)(2)	⋯	(n-m)(m)	⋯	0

$\Rightarrow$ need $\displaystyle\sum_{m=1}^{n} 2(m-1) + m(n-m)$

$$= 2\sum_{m=1}^{n}(m-1) + n\sum_{m=1}^{n} m - \sum_{m=1}^{n} m^2$$

$$= 2\left(\frac{n(n-1)}{2}\right) + n\left(\frac{n(n+1)}{2}\right) - \frac{n(n+1)(2n+1)}{6}$$

$\Rightarrow$ The leading order $= \dfrac{n^3}{2} - \dfrac{n^3}{3} = \dfrac{n^3}{6}$ #

Cholesky

Step 1 , set $l_{11} = \sqrt{a_{11}}$

Step 2. For $j = 2, \cdots n$, set $l_{j1} = a_{j1} / l_{11}$

Step 3, For $i = 2, \cdots, n-1$ do step4 and step5

Step 4 set $l_{ii} = \left(a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2 \right)^{1/2}$

Step 5 For $j = i+1, \cdots, n$

set $l_{ji} = \left(a_{ji} - \sum_{k=1}^{i-1} l_{jk} l_{ik} \right) / l_{ii}$

Step 6 set $l_{nn} = \left(a_{nn} - \sum_{k=1}^{n-1} l_{nk}^2 \right)^{1/2}$

i	1	2 ...	m	n
step2	n-1	0	0	0
step4	0	1	m-1	0
step5	0	(n-2)2	(n-m)m	0
step6	0	0		n-1

total need $2(n-1) + \sum_{m=2}^{n-1} (m-1) + nm - m^2 + cn$

(c 為做一次更號的工作量)

$\Rightarrow$ leading order $= \frac{n^3}{6}$ #

3, if $A = LDL^T =$

$$= \left(\begin{array}{c|c} l_{11} & \\ \hline & \ddots \end{array}\right) \left(\begin{array}{c|c} d_1 & 0 \\ \hline 0 & \begin{matrix} \ddots & 0 \\ 0 & d_{nn} \end{matrix} \end{array}\right) \left(\begin{array}{c|c} l_{11} & \\ \hline & \ddots \end{array}\right)$$

$$= \left(\begin{array}{c|c} d_1 & d_1 \cdot L(2{:}\mathrm{end},1)^T \\ \hline d_1 \cdot L(2{:}\mathrm{end},1) & \end{array}\right) , \quad l_{ii} = 1 \text{ for } 1 \le i \le n$$

$$\Rightarrow A - d_1 L(:,1) \cdot L(:,1)^T = \left(\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A^{(2)} & \\ 0 & & & \end{array}\right)_{n \times n}$$

$$\Rightarrow d_1 = a_{11} , \text{ and } L(:,1) = \frac{A(:,1)}{d_1}$$

and then we get a $(n-1) \times (n-1)$ matrix $A^{(2)}$

$$\Rightarrow \text{Let } d_2 = A^{(2)}(1,1), \quad L^{(2)}(:,1) = \frac{A^{(2)}(:,1)}{d_2}$$

$$\text{and } A^{(2)} - d_2 \cdot L^{(2)}(:,1) \cdot L^{(2)}(:,1)^T = \left(\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A^{(3)} & \\ 0 & & & \end{array}\right)_{(n-1) \times (n-1)}$$

$$\text{Let } L(:,2) = \begin{pmatrix} 0 \\ L^{(2)}(:,1) \end{pmatrix}$$

then find $L^{(i)}(:,1)$ and d_i , for $3 \le i \le n$

Hence

Input $A_{n\times n}$

$n = \text{length}(A(1,:))$

$L = \text{zeros}(n,n)$;

$d = \text{zeros}(1,n)$;

for $i = 1, \cdots\cdots n-1$

$d(i) = a_{ii}$;

$L(i:n, i) = A(i:n, i)/d(i)$

for $j = i+1, \cdots\cdots n$ % update A, i.e. comput $A^{(2)}, A^{(3)} \cdots\cdots$

$A(i+1:n, j) = A(i+1:n, j) - L(i+1:n, i)$

end

只更新 $A^{(i)}$ 的部份

，一次更新一行

end

$d(n) = A(n,n)$

$D = \text{diag}(d)$

output L, D

4.

$$A = \begin{pmatrix} \frac{-2}{h^2}-\alpha & \frac{1}{h^2} & & & \\ \frac{1}{h^2} & \frac{-2}{h^2}-\alpha & \frac{1}{h^2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{h^2} & \frac{-2}{h^2}-\alpha & \frac{1}{h^2} \\ & & & \frac{1}{h^2} & \frac{-2}{h^2}-\alpha \end{pmatrix} = \begin{pmatrix} -2N^2-\alpha & N^2 & & & O \\ N^2 & -2N^2-\alpha & N^2 & & \\ & \ddots & \ddots & \ddots & \\ O & & N^2 & -2N^2-\alpha & N^2 \\ & & & N^2 & -2N^2-\alpha \end{pmatrix}$$

(a) (1)

$\because |a_{ii}| = 2N^2+\alpha > \sum_{j\neq i} |a_{ij}| = 2N^2$

$\therefore$ A is strictly diagonally dominant

Thus by 6.19 $\Rightarrow$ Do Gaussian elimination without any row or column interchanges

(2) if use the method G-E with partial pivoting

$\because \max\{|a_{11}|, |a_{21}|\} = |a_{11}| = \frac{2}{h^2}+\alpha$

$\therefore$ In 1st step, we do G-E with partial pivoting without row or column interchanges

and by the proof in Thm 6.19, after (k-1)th step the new matrix $A^{(k)}$ is still strictly diagonally dominant

In $A^{(k)}$

$$\begin{pmatrix} \ddots & & \\ & \boxed{\begin{matrix} a^{(k)}_{k,k} & a^{(k)}_{k,k+1} \\ a^{(k)}_{k+1,k} & a^{(k)}_{k+1,k+1} \end{matrix}} & \\ & & a^{(k)}_{k+1,k+2} \end{pmatrix}$$

$\because a^{(k)}_{k,k} > a^{(k)}_{k,k+1} = a^{(k)}_{k+1,k}$ ($\because$ A是三對角 $\therefore a^{(k)}_{k,k+1} = a_{k,k+1}$ 不變)

$\therefore$ Don't any row interchanges

(3) scaled partial pivoting :

$$\begin{pmatrix} \ddots & & & & \\ & a_{m,m}^{(m)} & a_{m,m+1}^{(m)} & & \\ & a_{m+1,m}^{(m)} & a_{m+1,m+1}^{(m)} & a_{m+1,m+2}^{(m)} & \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix}$$

In mth step.

$\because$ $A^{(m)}$ is still strictly diagonally dominat

$\therefore$ $a_{m,m}^{(m)} > a_{m,m+1}^{(m)}$

$$\Rightarrow \frac{a_{m,m}^{(m)}}{|S_m|} > \frac{a_{m,m+1}^{(m)}}{|S_m|} = \frac{\frac{1}{h^2}}{\frac{2}{h^2}+\alpha} = \frac{a_{m+1,m}^{(m)}}{|S_{m+1}|}$$

So, In mth step, don't any row or column interchanges ※

(b)

$$A = \mathrm{diag}\left((-2N^2-\alpha)\times \mathrm{ones}(N-1,1)\right) + \mathrm{diag}\left(N^2 \times \mathrm{ones}(N-2,1),\ 1\right) + \mathrm{diag}\left(N^2 \times \mathrm{ones}(N-2,1),\ -1\right)$$

#

(c) look code !