HW 07

2. Let $h = \frac{1}{N}$, $x_j = jh = \frac{j}{N}$, $u(x_j) = u_j$, for $j = 0, 1, 2, \cdots, N$

$$u''(x_j) = \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2}, \quad \text{for } j = 1, 2, \cdots, N-1$$

$$u'(x_j) = \frac{u_{j+1} - u_{j-1}}{2h}, \quad \text{for } j = 1, 2, \cdots, N-1$$

$$\Rightarrow \underbrace{\frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & & 0 \\ 1 & -2 & 1 & 0 & & & \vdots \\ 0 & 1 & -2 & 1 & & & \\ \vdots & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & 0 \\ & & & 0 & 1 & -2 & 1 \\ 0 & \cdots & & 0 & 0 & 1 & -2 \end{bmatrix}}_{\overset{\text{Let}}{\|} \, A} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix}}_{\| \, \vec{u}} + \begin{bmatrix} u_0/h^2 = \alpha/h^2 \\ 0 \\ \vdots \\ 0 \\ -u_N/h^2 = -\beta/h^2 \end{bmatrix} = \begin{bmatrix} u''(x_1) \\ u''(x_2) \\ \vdots \\ u''(x_{N-1}) \end{bmatrix}$$

$$\underbrace{\frac{1}{2h}\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & & 0 \\ -1 & 0 & 1 & 0 & & & \vdots \\ 0 & -1 & 0 & 1 & & & \\ \vdots & & \ddots & \ddots & \ddots & & \\ & & & -1 & 0 & 1 & 0 \\ & & & 0 & -1 & 0 & 1 \\ 0 & \cdots & & 0 & 0 & -1 & 0 \end{bmatrix}}_{\overset{\text{Let}}{\|} \, B} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix}}_{\| \, \vec{u}} + \begin{bmatrix} -u_0/2h = -\alpha/2h \\ 0 \\ \vdots \\ 0 \\ u_N/2h = \beta/2h \end{bmatrix} = \begin{bmatrix} u'(x_1) \\ u'(x_2) \\ \vdots \\ u'(x_{N-1}) \end{bmatrix}$$

$$\Rightarrow (A+B)\vec{u} + \begin{bmatrix} \alpha(\frac{1}{h^2} - \frac{1}{2h}) \\ 0 \\ \vdots \\ 0 \\ \beta(\frac{1}{2h} - \frac{1}{h^2}) \end{bmatrix} = \begin{bmatrix} u''(x_1) + u'(x_1) \\ u''(x_2) + u'(x_2) \\ \vdots \\ u''(x_{N-1}) + u'(x_{N-1}) \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \end{bmatrix}$$

3.

$$[A \mid b] = \left[\begin{array}{cccccccccc|c} a_{11} & a_{12} & a_{13} & 0 & \cdots & & & & & 0 & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 & \cdots & & & & 0 & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & 0 & \cdots & & & 0 & b_3 \\ 0 & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & 0 & \cdots & & 0 & b_4 \\ \vdots & & \ddots & & \ddots & & \ddots & & \ddots & \vdots & \vdots \\ & & & & & & & & & 0 & \\ & & & & & & & & & a_{N^2-2,N^2} & \\ & & & & & & & & & a_{N^2-1,N^2} & \\ 0 & \cdots & & & & & 0 & a_{N^2,N^2-2} & a_{N^2,N^2-1} & a_{N^2,N^2} & b_{N^2} \end{array}\right] = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_{N^2} \end{bmatrix} = \begin{bmatrix} E_1^{(1)} \\ E_2^{(1)} \\ \vdots \\ E_{N^2}^{(1)} \end{bmatrix}$$

$$[A \mid b] = [A^{(1)} \mid b^{(1)}]$$

① Use $E_1^{(1)}$ to eliminate x_1 from the equation $E_2^{(1)}, E_3^{(1)}$,

we need compute $m_{12} = \frac{a_{21}^{(1)}}{a_{11}^{(1)}}$ need 1 multip.

$m_{13} = \frac{a_{31}^{(1)}}{a_{11}^{(1)}}$ need 1 multip.

and $(E_2^{(1)} - m_{12} E_1^{(1)}) \to E_2^{(2)}$ need 3 multip.

$(E_3^{(1)} - m_{13} E_1^{(1)}) \to E_3^{(2)}$ need 3 multip.

So, total unmber of multiplications is $2+3+3=8$

$\Rightarrow$ Use $E_i^{(i)}$ to eliminate x_i from the equation $E_{i+1}^{(i)}, E_{i+2}^{(i)}$,

we need compute $m_{i,i+1}$ need 1 multip.

$m_{i,i+2}$ need 1 multip.

and $(E_{i+1}^{(i)} - m_{i,i+1} E_i^{(i)}) \to E_{(i+1)}^{(i+1)}$ need 3 multip.

$(E_{i+2}^{(i)} - m_{i,i+2} E_i^{(i)}) \to E_{(i+2)}^{(i+1)}$ need 3 multip.

for $i = 1, 2, \cdots N^2-2$.

& for $i = N^2-1$:

we need compute $m_{N^2-1,N^2} = \dfrac{a^{(N^2-1)}_{N^2,N^2-1}}{a^{(N^2-1)}_{N^2-1,N^2-1}}$ need 1 multip.

and $(E_{N^2}^{(N^2-1)} - m_{N^2-1,N^2} E_{N^2-1}^{(N^2-1)}) \to E_{N^2}^{(N^2)}$ we need 2 multip.

So.

$$\left[A \,\middle|\, b\right] \rightarrow \left[\begin{array}{cccccc|c} a_{11}^{(N^2)} & a_{12}^{(N^2)} & a_{13}^{(N^2)} & 0 & \cdots & 0 & b_1^{(N^2)} \\ & \ddots & \ddots & \ddots & & \vdots & b_2^{(N^2)} \\ & & \ddots & \ddots & \ddots & 0 & \vdots \\ & & & \ddots & \ddots & a_{N^2-2,N^2}^{(N^2)} & \vdots \\ & & & & \ddots & a_{N^2-1,N^2}^{(N^2)} & \vdots \\ & & & & & a_{N^2N^2}^{(N^2)} & b_{N^2}^{(N^2)} \end{array}\right] = \left[A^{(N^2)} \,\middle|\, b^{(N^2)}\right]$$

we need $(N^2-2)\times 8+3 = 8N^2-13$

② Use $E_{N^2}^{(N^2)}$ to eliminate x_{N^2} from the equation $E_{N^2-1}^{(N^2)}, E_{N^2-2}^{(N^2)}$

we need compute m_{N^2,N^2-1} need 1 multip.

m_{N^2,N^2-2} need 1 multip.

and $(E_{N^2-1}^{(N^2)} - m_{N^2,N^2-1}E_{N^2}^{(N^2)}) \rightarrow E_{N^2-1}^{(N^2+1)}$ need 1 multip.

$(E_{N^2-2}^{(N^2)} - m_{N^2,N^2-2}E_{N^2}^{(N^2)}) \rightarrow E_{N^2-2}^{(N^2+1)}$ need 1 multip.

$$\Rightarrow \left[A^{(N^2)} \,\middle|\, b^{(N^2)}\right] \rightarrow \left[\begin{array}{ccc|c} a_{11}^{(2N^2)} & & 0 & b_1^{(2N^2)} \\ & \ddots & & \vdots \\ 0 & & a_{N^2N^2}^{(2N^2)} & b_{N^2}^{(2N^2)} \end{array}\right]$$

we need $(N^2-1)(4)+2 = 4N^2-6$

③

compute $x_i = \dfrac{b_i^{(2N^2)}}{a_{ii}^{(2N^2)}}$, $1\le i\le N^2$

we need N^2 multip.

By ①②③ we need $8N^2-13+4N^2-6+N^2 = 13N^2-19$

#

4-1

$$[A \mid b] = \left[\begin{array}{ccccc|c} a_{11} & \cdots & a_{1,N+1} & & 0 & b_1 \\ \vdots & \ddots & & \ddots & & b_2 \\ a_{N+1,1} & & \ddots & & \ddots & \vdots \\ & \ddots & & \ddots & a_{N^2-N,N^2} & \vdots \\ & & \ddots & & \vdots & \vdots \\ 0 & & & a_{N^2,N^2-N} & \cdots\ a_{N^2,N^2} & b_{N^2} \end{array}\right] = [A^{(1)} \mid b^{(1)}]$$

① use $E_i^{(i)}$ to eliminate x_i from the equation $E_{i+1}^{(i)} \cdots E_{i+N}^{(i)}$

need N multiplications to compute m_{ij}, $i+1 \le j \le i+N$

and
$$\left.\begin{array}{lll} (E_{i+1}^{(i)} - m_{i,i+1}E_i^{(i)}) \to E_{i+1}^{(i+1)} & \text{need} & N+1 \\ (E_{i+2}^{(i)} - m_{i,i+2}E_i^{(i)}) \to E_{i+2}^{(i+1)} & \text{need} & N+1 \\ \vdots & & \vdots \\ (E_{i+N}^{(i)} - m_{i,i+N}E_i^{(i)}) \to E_{i+N}^{(i+1)} & \text{need} & N+1 \end{array}\right\} \text{total } N(N+1)$$

for $1 \le i \le N^2-N$

So. $$[A^{(1)} \mid b^{(1)}] \to [A^{(N^2-N+1)} \mid b^{(N^2-N+1)}] = \left[\begin{array}{cccc|c} x & \cdots & x & 0 & b_1^{(N^2-N+1)} \\ 0 & \ddots & x\ \cdots & x & \vdots \\ 0 & & \ddots & \vdots & \vdots \\ & & x & \cdots\ x & b_{N^2}^{(N^2-N+1)} \end{array}\right]$$

we need $(N^2-N)(N)(N+1)$ multiplications

②

$$\left[A^{(N^2-N+1)} \,\middle|\, b^{(N^2-N+1)}\right] = \left[\begin{array}{cccc|c} x & \cdots & x & & \\ & x & \cdots & x & b^{(N^2-N+1)} \\ & \vdots & \ddots & \vdots & \\ & x & \cdots & x & \end{array}\right] \begin{array}{l} \leftarrow E_{N^2-N+1}^{(N^2-N+1)} \\ \\ \leftarrow E_{N^2}^{(N^2-N+1)} \end{array}$$

we focus on nxn submatrix that every entries are nonzero

$\Rightarrow$ we need $\dfrac{2N^3+3N^2-5N}{6}$ multip. such that

$$\left[A^{(N^2-N+1)} \,\middle|\, b^{(N^2-N+1)}\right] \rightarrow \left[\begin{array}{ccc|c} x & \cdots & x & \\ & \ddots & \vdots & b^{(N^2)} \\ 0 & & x & \end{array}\right] = \left[A^{(N^2)} \,\middle|\, b^{(N^2)}\right]$$

③ $\left[A^{(N^2)} \,\middle|\, b^{(N^2)}\right] \rightarrow \left[A^{(2N^2)} \,\middle|\, b^{(2N^2)}\right]$

where $A^{(2N^2)}$ is diagonal matrix

we need $(2N)(N^2-N) + 2\sum_{j=1}^{N-1}(N-j)$

④ need N^2 to compute solution $x_i = \dfrac{b^{(2N^2)}}{a_{ii}^{(2N^2)}}$

by ①②③④, total $= (N^2-N)(N)(N+1) + \dfrac{2N^3+3N^2-5N}{6}$

$$+ (2N)(N^2-N) + 2\sum_{j=1}^{N-1}(N-j) + N^2$$

$$= N^4 + \cdots\cdots$$

5.

$$[A|b] = \left[\begin{array}{cccccccccc|c} a_{11} & a_{12} & 0 & \cdots & 0 & a_{1,N+1} & 0 & \cdots & & 0 & b_1 \\ a_{21} & a_{22} & a_{23} & 0 & \cdots & 0 & a_{2,N+2} & & & & b_2 \\ 0 & a_{32} & a_{33} & a_{34} & & & & a_{3,N+3} & & 0 & b_3 \\ \vdots & & & & \ddots & & & & & & \vdots \\ 0 & & & & & & & & & & \\ a_{N+1,1} & 0 & & & & & & & & 0 & \\ 0 & a_{N+2,1} & 0 & & & & & & & a_{N^2-N,N^2} & \\ & 0 & a_{N+3,1} & & & & & & & 0 & \\ \vdots & & & \ddots & & & & & & \vdots & \\ & & & & 0 & & & & & 0 & \\ & & 0 & & & & & & & a_{N^2-1,N^2} & b_{N^2-1} \\ 0 & & \cdots & & 0 & a_{N^2,N^2-N} & 0 & \cdots & 0 \; a_{N^2,N^2-1} & a_{N^2N^2} & b_{N^2} \end{array}\right] = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ \\ E_{N^2} \end{bmatrix}$$

$$= [A^{(1)} | b^{(1)}] = \begin{bmatrix} E_1^{(1)} \\ \vdots \\ E_{N^2}^{(1)} \end{bmatrix}$$

①

use $E_1^{(1)}$ to eliminate x_1 from the equation $E_2^{(1)}$, $E_{N+1}^{(1)}$

we need $2+3+3 = 2+2\times3 = 2(3+1)$ multip.

and $(E_2^{(1)} - m_{12}E_1^{(1)}) \to E_2^{(2)}$, $(E_{N+1}^{(1)} - m_{1,N+1}E_1^{(1)}) \to E_{N+1}^{(2)}$

$\Rightarrow \quad E_2^{(2)}(N+1) \neq 0 \quad , \quad E_{N+1}^{(2)}(2) \neq 0$

use $E_2^{(2)}$ to eliminate x_2 from the equation $E_3^{(2)}$, $\underline{E_{N+1}^{(2)}}$, $E_{N+2}^{(2)}$

$\because$ the row 2 of $A^{(2)}$ has 4 nonzero entries

$\therefore$ we need $3 + 3\times4 = 3(4+1)$ multip.

$\vdots$

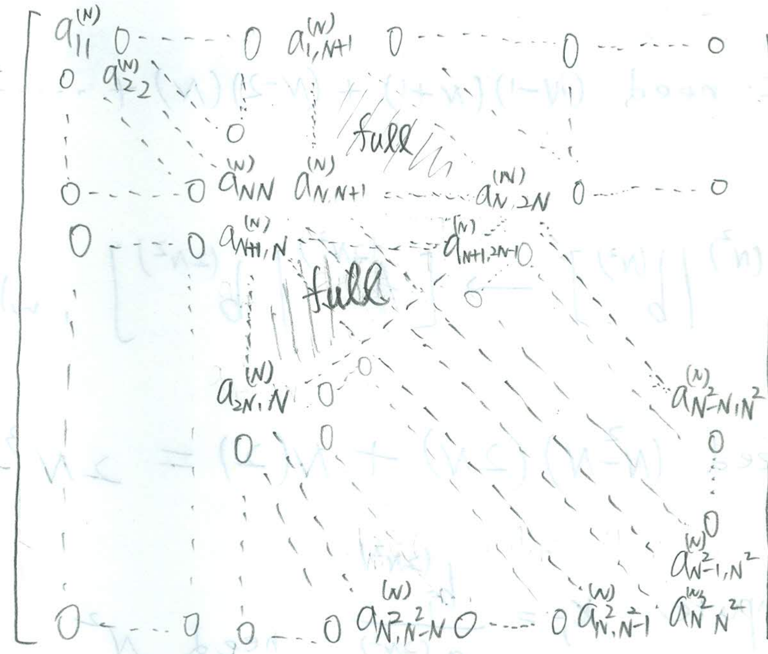
use $E_{N-1}^{(N-1)}$ to eliminate x_{N-1} from the equation $E_N^{(N-1)}, E_{N+1}^{(N-1)}, \ldots, E_{2N-1}^{(N-1)}$

$\because$ the row $N-1$ of $A^{(N-1)}$ has N nonzero entries

$\therefore$ need $N-1+(N-1)N = (N-1)(N+1)$

③

Now, $\left[A^{(N)} \mid b^{(N)}\right] =$



$\because$ the row N of $A^{(N)}$ has $N+1$ nonzero entries

and use $E_N^{(N)}$ to eliminate x_N from the equation

$\underbrace{E_{N+1}^{(N)}, E_{N+2}^{(N)}, \ldots, E_{2N}^{(N)}}_{N}$ we need $N+(N+1)\cdot N$

$= (N+2)N$

untill, use $E_{N^2-N}^{(N^2-N)}$ to eliminate x_{N^2-N} from the equation

$\underbrace{E_{N^2-N+1}^{(N^2-N)}, E_{N^2-N+2}^{(N^2-N)}, \ldots\ldots E_{N^2}^{(N)}}_{N}$, we need $N+(N+1)\cdot N$

$= (N+2)N$

③ use $E^{(N^2-N+1)}_{N^2-N+1}$ to eliminate x_{N^2-N+1} from the equation

$$\underbrace{E^{(N^2-N+1)}_{N^2-N+2},\ E^{(N^2-N+1)}_{N^2-N+3},\ \cdots\cdots,\ E^{(N^2-N+1)}_{N^2}}$$

need $N-1+(N-1)N = (N-1)(N+1)$

untill $\to [A^{(N^2)} | b^{(N^2)}]$, where $A^{(N^2)}$ is upper triangular matrix

It need $(N-1)(N+1)+(N-2)(N)+\cdots\cdots+(1)(3)$

④ $[A^{(N^2)} | b^{(N^2)}] \longrightarrow [A^{(2N^2)} | b^{(2N^2)}]$, where $A^{(2N^2)}$ is diagonal matrix

$\Rightarrow$ we need $(N^2-N)(2N)+N(2) = 2N^3-2N^2+2N$

⑤ compute $x_i = \dfrac{b_i^{(2N^2)}}{a_{ii}^{(2N^2)}}$ need N^2

By ①②③④⑤ total: $2\times4+3\times5+\cdots+(N-1)(N+1)$

$+(N^2-2N+1)(N)(N+2)$

$+(N-1)(N+1)+(N-2)(N)+\cdots+(1)(3)$

$+2N^3-2N^2+2N+N^2$

$= \underline{N^4} + \cdots\cdots$

※