The numbers in the table are given to 6 decimal places, so it is reasonable to let $\varepsilon = 0.0000005$. The optimal value of h is

$$h = 2\sqrt{\frac{\varepsilon}{M}} = 2\sqrt{\frac{0.0000005}{37.9050567}} = 0.000229703.$$

25. The three-point formulas give the results in the following table.

Time	0	3	5	8	10	13
Speed	79	82.4	74.2	76.8	69.4	71.2

26. The three-point formulas give the results in the following table.

t	1.00	1.01	1.02	1.03	1.04
$\varepsilon(t)$	2.400	2.403	3.386	5.352	7.320

27. The approximations eventually become zero since the numerator becomes zero.

28. By averaging the Taylor polynomials we have

$$f'''(x_0) = \frac{1}{h^3}\left[-\frac{1}{2}f(x_0-2h) + f(x_0-h) - f(x_0+h) + \frac{1}{2}f(x_0+2h)\right] - \frac{h^2}{4}f^{(5)}(\xi),$$

where ξ is between $x_0 - 2h$ and $x_0 + 2h$.

29. Since $e'(h) = -\varepsilon/h^2 + hM/3$, we have $e'(h) = 0$ if and only if $h = \sqrt[3]{3\varepsilon/M}$. Also, $e'(h) < 0$ if $h < \sqrt[3]{3\varepsilon/M}$ and $e'(h) > 0$ if $h > \sqrt[3]{3\varepsilon/M}$, so an absolute minimum for $e(h)$ occurs at $h = \sqrt[3]{3\varepsilon/M}$.

Exercise Set 4.2, page 184

1. (a) $f'(1) \approx 1.0000109$ (b) $f'(0) \approx 2.0000000$

   (c) $f'(1.05) \approx 2.2751459$ (d) $f'(2.3) \approx -19.646799$

2. (a) $f'(1) \approx 0.99999998$ (b) $f'(0) \approx 1.9999999$

   (c) $f'(1.05) \approx 2.2751458$ (d) $f'(2.3) \approx -19.646796$

3. (a) $f'(1) \approx 1.001$ (b) $f'(0) \approx 1.999$

(c) $f'(1.05) \approx 2.283$ (d) $f'(2.3) \approx -19.61$

4. (a) $f'(1) \approx 0.9999$ (b) $f'(0) \approx 1.997$

   (c) $f'(1.05) \approx 2.282$ (d) $f'(2.3) \approx -19.66$

5. $\int_0^{\pi} \sin x \, dx \approx 1.999999$

6. $\int_0^{3\pi/2} \cos x \, dx \approx -1.000135$

7. With $h = 0.1$, Formula (4.6) becomes

$$f'(2) \approx \frac{1}{1.2}\left[1.8e^{1.8} - 8\left(1.9e^{1.9}\right) + 8(2.1)e^{2.1} - 2.2e^{2.2}\right] = 22.166995.$$

With $h = 0.05$, Formula (4.6) becomes

$$f'(2) \approx \frac{1}{0.6}\left[1.9e^{1.9} - 8\left(1.95e^{1.95}\right) + 8(2.05)e^{2.05} - 2.1e^{2.1}\right] = 22.167157.$$

8. The formula $f'(x_0) = \frac{1}{12h}\left[f(x_0+4h) - 12f(x_0+2h) + 32f(x_0+h) - 21f(x_0)\right]$ is $O\left(h^3\right)$.

9. Let

$$N_2(h) = N\left(\frac{h}{3}\right) + \left(\frac{N\left(\frac{h}{3}\right) - N(h)}{2}\right) \quad \text{and} \quad N_3(h) = N_2\left(\frac{h}{3}\right) + \left(\frac{N_2\left(\frac{h}{3}\right) - N_2(h)}{8}\right).$$

Then $N_3(h)$ is an $O(h^3)$ approximation to M.

10. Let $N_2(h) = N\left(\frac{h}{3}\right) + \frac{1}{8}\left(N\left(\frac{h}{3}\right) - N(h)\right)$ and $N_3(h) = N_2\left(\frac{h}{3}\right) + \frac{1}{80}\left(N_2\left(\frac{h}{3}\right) - N_2(h)\right)$. Then $N_3(h)$ is an $O\left(h^6\right)$ approximation to M.

11. Let $N(h) = (1+h)^{1/h}$, $N_2(h) = 2N\left(\frac{h}{2}\right) - N(h)$, $N_3(h) = N_2\left(\frac{h}{2}\right) + \frac{1}{3}(N_2\left(\frac{h}{2}\right) - N_2(h))$.

   (a) $N(0.04) = 2.665836331$, $N(0.02) = 2.691588029$, $N(0.01) = 2.704813829$

   (b) $N_2(0.04) = 2.717339727$, $N_2(0.02) = 2.718039629$. The $O(h^3)$ approximation is $N_3(0.04) = 2.718272931$.

   (c) Yes, since the errors seem proportioned to h for $N(h)$, to h^2 for $N_2(h)$, and to h^3 for $N_3(h)$.

12. (a) We have

$$\lim_{h\to 0} \frac{\ln(2+h) - \ln(2-h)}{h} = \lim_{h\to 0} \frac{1}{2+h} + \frac{1}{2-h} = 1,$$

so

$$\lim_{h\to 0} \left(\frac{2+h}{2-h}\right)^{1/h} = \lim_{h\to 0} e^{\frac{1}{h}[\ln(2+h) - \ln(2-h)]} = e^1 = e.$$

(b) $N(0.04) = 2.718644377221219$, $N(0.02) = 2.718372444800607$, $N(0.01) = 2.718304481241685$

(c) Let $N_2(h) = 2N\left(\frac{h}{2}\right) - N(h)$ and $N_3(h) = N_2\left(\frac{h}{2}\right) + \frac{1}{3}\left[N_2\left(\frac{h}{2}\right) - N_2(h)\right]$. Then $N_2(0.04) = 2.718100512379995$, $N_2(0.02) = 2.718236517682763$ and $N_3(0.04) = 2.718281852783685$. $N_3(0.04)$ is an $O\left(h^3\right)$ approximation satisfying $|e - N_3(0.04)| \leq 0.5 \times 10^{-7}$.

(d)

$$N(-h) = \left(\frac{2-h}{2+h}\right)^{1/-h} = \left(\frac{2+h}{2-h}\right)^{1/h} = N(h)$$

(e) Let

$$e = N(h) + K_1 h + K_2 h^2 + K_3 h^3 + \cdots.$$

Replacing h by $-h$ gives

$$e = N(-h) - K_1 h + K_2 h^2 - K_3 h^3 + \cdots,$$

but $N(-h) = N(h)$, so that

$$e = N(h) - K_1 h + K_2 h^2 - K_3 h^3 + \cdots.$$

Thus,

$$K_1 h + K_3 h^3 + \cdots = -K_1 h - K_3 h^3 \cdots,$$

and it follows that $K_1 = K_3 = K_5 = \cdots = 0$ and

$$e = N(h) + K_2 h^2 + K_4 h^4 + \cdots.$$

(f) Let

$$N_2(h) = N\left(\frac{h}{2}\right) + \frac{1}{3}\left(N\left(\frac{h}{2}\right) - N(h)\right)$$

and

$$N_3(h) = N_2\left(\frac{h}{2}\right) + \frac{1}{15}\left(N_2\left(\frac{h}{2}\right) - N_2(h)\right).$$

Then

$$N_2(0.04) = 2.718281800660402, N_2(0.02) = 2.718281826722043$$

and

$$N_3(0.04) = 2.718281828459487.$$

$N_3(0.04)$ is an $O\left(h^6\right)$ approximation satisfying

$$|e - N_3(0.04)| \leq 0.5 \times 10^{-12}.$$

13. (a) We have

$$P_{0,1}(x) = \frac{\left(x - h^2\right) N_1\left(\frac{h}{2}\right)}{\frac{h^2}{4} - h^2} + \frac{\left(x - \frac{h^2}{4}\right) N_1(h)}{h^2 - \frac{h^2}{4}}, \quad \text{so} \quad P_{0,1}(0) = \frac{4N_1\left(\frac{h}{2}\right) - N_1(h)}{3}.$$

Similarly,

$$P_{1,2}(0) = \frac{4N_1\left(\frac{h}{4}\right) - N_1\left(\frac{h}{2}\right)}{3}.$$

(b) We have

$$P_{0,2}(x) = \frac{\left(x - h^4\right) N_2\left(\frac{h}{2}\right)}{\frac{h^4}{16} - h^4} + \frac{\left(x - \frac{h^4}{16}\right) N_2(h)}{h^4 - \frac{h^4}{16}}, \quad \text{so} \quad P_{0,2}(0) = \frac{16N_2\left(\frac{h}{2}\right) - N_2(h)}{15}.$$

14. All the approximations of the form $N_{2i}\left(h/2^j\right)$, for $i = 1, 2, \ldots$ and $j = 0, 1, 2, \ldots$, will be upper bounds for M, and all the approximations of the form $N_{2i+1}\left(\frac{h}{2^j}\right)$, for $i = 0, 1, 2, \ldots$ and $j = 0, 1, 2, \ldots$, will be lower bounds for M.

15. (a) The polygonal approximations are in the following table.

k	4	8	16	32	64	128	256	512
p_k	$2\sqrt{2}$	3.0614675	3.1214452	3.1365485	3.1403312	3.1412723	3.1415138	3.1415729
P_k	4	3.3137085	3.1825979	3.1517249	3.144184	3.1422236	3.1417504	3.1416321

(b) Values of p_k and P_k are given in the following tables, together with the extrapolation results:

For p_k we have :

2.8284271				
3.0614675	3.1391476			
3.1214452	3.1414377	3.1415904		
3.1365485	3.1415829	3.1415926	3.1415927	
3.1403312	3.1415921	3.1415927	3.1415927	3.1415927

For P_k we have :

4				
3.3137085	3.0849447			
3.1825979	3.1388943	3.1424910		
3.1517249	3.1414339	3.1416032	3.1415891	
3.1441184	3.1415829	3.1415928	3.1415926	3.1415927

Exercise Set 4.3, page 195

1. The Trapezoidal rule gives the following approximations.

(a) 0.265625 (b) −0.2678571 (c) −0.17776434 (d) 0.1839397

(e) −0.8666667 (f) −0.1777643 (g) 0.2180895 (h) 4.1432597