

# Numerical Differentiation and Integration

## Exercise Set 4.1, page 176

1. From the forward-backward difference formula (4.1), we have the following approximations:

(a)  $f'(0.5) \approx 0.8520$ ,  $f'(0.6) \approx 0.8520$ ,  $f'(0.7) \approx 0.7960$

(b)  $f'(0.0) \approx 3.7070$ ,  $f'(0.2) \approx 3.1520$ ,  $f'(0.4) \approx 3.1520$

2. The approximations are in the following tables

(a)

| $x$  | $f(x)$ | $f'(x)$ |
|------|--------|---------|
| -0.3 | 1.9507 | 0.9140  |
| -0.2 | 2.0421 | 0.9140  |
| -0.1 | 2.0601 | 0.1800  |

(b)

| $x$ | $f(x)$ | $f'(x)$ |
|-----|--------|---------|
| 1.0 | 1.0000 | 1.3125  |
| 1.2 | 1.2625 | 1.3125  |
| 1.4 | 1.6595 | 1.9850  |

3. The approximations are in the following tables.

(a)

| $x$ | Actual Error | Error Bound |
|-----|--------------|-------------|
| 0.5 | 0.0255       | 0.0282      |
| 0.6 | 0.0267       | 0.0282      |
| 0.7 | 0.0312       | 0.0322      |

(b)

| $x$ | Actual Error | Error Bound |
|-----|--------------|-------------|
| 0.0 | 0.2930       | 0.3000      |
| 0.2 | 0.2694       | 0.2779      |
| 0.4 | 0.2602       | 0.2779      |

4. (a)

| $x$  | Actual Error | Error Bound |
|------|--------------|-------------|
| -0.3 | 0.34457      | 0.36842     |
| -0.2 | 0.35633      | 0.36842     |
| -0.1 | 0.38533      | 0.39203     |

(b)

| $x$ | Actual Error | Error Bound |
|-----|--------------|-------------|
| 1.0 | 0.31250      | 0.33646     |
| 1.2 | 0.32507      | 0.33646     |
| 1.4 | 0.35712      | 0.36729     |

5. For the endpoints of the tables, we use Formula (4.4). The other approximations come from Formula (4.5).
- (a)  $f'(1.1) \approx 17.769705$ ,  $f'(1.2) \approx 22.193635$ ,  $f'(1.3) \approx 27.107350$ ,  $f'(1.4) \approx 32.150850$   
 (b)  $f'(8.1) \approx 3.092050$ ,  $f'(8.3) \approx 3.116150$ ,  $f'(8.5) \approx 3.139975$ ,  $f'(8.7) \approx 3.163525$   
 (c)  $f'(2.9) \approx 5.101375$ ,  $f'(3.0) \approx 6.654785$ ,  $f'(3.1) \approx 8.216330$ ,  $f'(3.2) \approx 9.786010$   
 (d)  $f'(2.0) \approx 0.13533150$ ,  $f'(2.1) \approx -0.09989550$ ,  $f'(2.2) \approx -0.3298960$ ,  $f'(2.3) \approx -0.5546700$
6. For the endpoints of the tables, we use Formula (4.4). The other approximations come from Formula (4.5).

(a)

| $x$  | $f(x)$   | $f'(x)$  |
|------|----------|----------|
| -0.3 | -0.27652 | -0.06030 |
| -0.2 | -0.25074 | 0.57590  |
| -0.1 | -0.16134 | 1.25370  |
| 0.0  | 0.0      | 1.97310  |

(b)

| $x$ | $f(x)$   | $f'(x)$  |
|-----|----------|----------|
| 7.4 | -68.3193 | -16.6933 |
| 7.6 | -71.6982 | -17.0958 |
| 7.8 | -75.1576 | -17.4980 |
| 8.0 | -78.6974 | -17.9000 |

(c)

| $x$ | $f(x)$  | $f'(x)$  |
|-----|---------|----------|
| 1.1 | 1.52918 | 1.34360  |
| 1.2 | 1.64024 | 0.87760  |
| 1.3 | 1.70470 | 0.36265  |
| 1.4 | 1.71277 | -0.20125 |

(d)

| $x$  | $f(x)$   | $f'(x)$   |
|------|----------|-----------|
| -2.7 | 0.054797 | -0.915178 |
| -2.5 | 0.11342  | 1.50141   |
| -2.3 | 0.65536  | 2.17825   |
| -2.1 | 0.98472  | 1.11535   |

7. The errors and error bounds are given in the following tables.

| (a) | $x$ | Actual Error | Error Bound | (b) | $x$ | Actual Error           | Error Bound |
|-----|-----|--------------|-------------|-----|-----|------------------------|-------------|
|     | 1.1 | 0.280322     | 0.359033    |     | 8.1 | 0.00018594             | 0.000020322 |
|     | 1.2 | 0.147282     | 0.179517    |     | 8.3 | 0.00010551             | 0.000010161 |
|     | 1.3 | 0.179874     | 0.219262    |     | 8.5 | $9.116 \times 10^{-5}$ | 0.000009677 |
|     | 1.4 | 0.378444     | 0.438524    |     | 8.7 | 0.00020197             | 0.000019355 |

| (c) | $x$ | Actual Error | Error Bound | (d) | $x$ | Actual Error | Error Bound |
|-----|-----|--------------|-------------|-----|-----|--------------|-------------|
|     | 2.9 | 0.011956     | 0.0180988   |     | 2.0 | 0.00252235   | 0.00410304  |
|     | 3.0 | 0.0049251    | 0.00904938  |     | 2.1 | 0.00142882   | 0.00205152  |
|     | 3.1 | 0.0004765    | 0.00493920  |     | 2.2 | 0.00204851   | 0.00260034  |
|     | 3.2 | 0.0013745    | 0.00987840  |     | 2.3 | 0.00437954   | 0.00520068  |

| 8. (a) | $x$  | Actual Error | Error Bound | (b) | $x$ | Actual Error | Error Bound |
|--------|------|--------------|-------------|-----|-----|--------------|-------------|
|        | -0.3 | 0.028638     | 0.029692    |     | 7.4 | 0.000367     | 0.000032    |
|        | -0.2 | 0.014097     | 0.014846    |     | 7.6 | 0.000083     | 0.000016    |
|        | -0.1 | 0.013577     | 0.014130    |     | 7.8 | 0.000041     | 0.000015    |
|        | 0.0  | 0.026900     | 0.028260    |     | 8.0 | 0.000000     | 0.000030    |

| (c) | $x$ | Actual Error | Error Bound | (d) | $x$  | Actual Error | Error Bound |
|-----|-----|--------------|-------------|-----|------|--------------|-------------|
|     | 1.1 | 0.033886     | 0.034784    |     | -2.7 | 0.511122     | 1.440958    |
|     | 1.2 | 0.016791     | 0.017392    |     | -2.5 | 0.435980     | 0.720479    |
|     | 1.3 | 0.015740     | 0.016817    |     | -2.3 | 0.632733     | 0.720479    |
|     | 1.4 | 0.030920     | 0.033633    |     | -2.1 | 1.044472     | 1.440958    |

9. The approximations and the formulas used are:

- (a)  $f'(2.1) \approx 3.899344$  from (4.7)  $f'(2.2) \approx 2.876876$  from (4.7)  $f'(2.3) \approx 2.249704$  from (4.6)  $f'(2.4) \approx 1.837756$  from (4.6)  $f'(2.5) \approx 1.544210$  from (4.7)  $f'(2.6) \approx 1.355496$  from (4.7)
- (b)  $f'(-3.0) \approx -5.877358$  from (4.7)  $f'(-2.8) \approx -5.468933$  from (4.7)  $f'(-2.6) \approx -5.059884$  from (4.6)  $f'(-2.4) \approx -4.650223$  from (4.6)  $f'(-2.2) \approx -4.239911$  from (4.7)  $f'(-2.0) \approx -3.828853$  from (4.7)

10. The approximations are in the following tables.

| (a) |      |            |          | (b) |      |          |           |
|-----|------|------------|----------|-----|------|----------|-----------|
|     | $x$  | $f(x)$     | $f'(x)$  |     | $x$  | $f(x)$   | $f'(x)$   |
|     | 1.05 | -1.709847  | 7.798690 |     | -3.0 | 16.08554 | -19.08087 |
|     | 1.10 | -1.373823  | 5.753747 |     | -2.8 | 12.64465 | -15.44088 |
|     | 1.15 | -1.119214  | 4.499409 |     | -2.6 | 9.863738 | -12.46303 |
|     | 1.20 | -0.9160143 | 3.675512 |     | -2.4 | 7.623176 | -10.02259 |
|     | 1.25 | -0.7470223 | 3.088414 |     | -2.2 | 5.825013 | -8.02097  |
|     | 1.30 | -0.6015966 | 2.710997 |     | -2.0 | 4.389056 | -6.38573  |

11. The approximations are in the following tables.

| (a) |     |              |             | (b) |      |                       |                       |
|-----|-----|--------------|-------------|-----|------|-----------------------|-----------------------|
|     | $x$ | Actual Error | Error Bound |     | $x$  | Actual Error          | Error Bound           |
|     | 2.1 | 0.0242312    | 0.109271    |     | -3.0 | $1.55 \times 10^{-5}$ | $6.33 \times 10^{-7}$ |
|     | 2.2 | 0.0105138    | 0.0386885   |     | -2.8 | $1.32 \times 10^{-5}$ | $6.76 \times 10^{-7}$ |
|     | 2.3 | 0.0029352    | 0.0182120   |     | -2.6 | $7.95 \times 10^{-7}$ | $1.05 \times 10^{-7}$ |
|     | 2.4 | 0.0013262    | 0.00644808  |     | -2.4 | $6.79 \times 10^{-7}$ | $1.13 \times 10^{-7}$ |
|     | 2.5 | 0.0138323    | 0.109271    |     | -2.2 | $1.28 \times 10^{-5}$ | $6.76 \times 10^{-7}$ |
|     | 2.6 | 0.0064225    | 0.0386885   |     | -2.0 | $7.96 \times 10^{-6}$ | $6.76 \times 10^{-7}$ |

|     |     |      |              |             |     |      |              |             |
|-----|-----|------|--------------|-------------|-----|------|--------------|-------------|
| 12. | (a) |      |              |             | (b) |      |              |             |
|     |     | $x$  | Actual Error | Error Bound |     | $x$  | Actual Error | Error Bound |
|     |     | 1.05 | 0.0484600    | 0.2185438   |     | -3.0 | 0.004666     | 0.006427    |
|     |     | 1.10 | 0.0210325    | 0.0773769   |     | -2.8 | 0.003763     | 0.005262    |
|     |     | 1.15 | 0.0058693    | 0.0364240   |     | -2.6 | 0.000711     | 0.001071    |
|     |     | 1.20 | 0.0026524    | 0.0128962   |     | -2.4 | 0.000591     | 0.000877    |
|     |     | 1.25 | 0.0276704    | 0.2185438   |     | -2.2 | 0.004041     | 0.006427    |
|     |     | 1.30 | 0.0128401    | 0.0773769   |     | -2.0 | 0.003329     | 0.005262    |

13.  $f'(3) \approx \frac{1}{12}[f(1) - 8f(2) + 8f(4) - f(5)] = 0.21062$ , with an error bound given by

$$\max_{1 \leq x \leq 5} \frac{|f^{(5)}(x)|h^4}{30} \leq \frac{23}{30} = 0.7\bar{6}.$$

14.  $f'(3) \approx \frac{1}{2}[f(4) - f(2)] = 0.21210$ , with an error bound given by

$$\max_{1 \leq x \leq 5} \frac{|f'''(x)| h^2}{6} \leq \frac{4}{2} = 0.6\bar{6}.$$

15. From the forward-backward difference formula (4.1), we have the following approximations:

(a)  $f'(0.5) \approx 0.852$ ,  $f'(0.6) \approx 0.852$ ,  $f'(0.7) \approx 0.7960$

(b)  $f'(0.0) \approx 3.707$ ,  $f'(0.2) \approx 3.153$ ,  $f'(0.4) \approx 3.153$

16. For the endpoints of the tables, we use Formula (4.7). The other approximations come from Formula (4.6).

(a)  $f'(1.1) \approx 17.75$ ,  $f'(1.2) \approx 22.17$ ,  $f'(1.3) \approx 27.10$ ,  $f'(1.4) \approx 32.50$ ,

(b)  $f'(8.1) \approx 3.075$ ,  $f'(8.3) \approx 3.125$ ,  $f'(8.5) \approx 3.150$ ,  $f'(8.7) \approx 3.150$ ,

(c)  $f'(2.9) \approx 5.080$ ,  $f'(3.0) \approx 6.655$ ,  $f'(3.1) \approx 8.220$ ,  $f'(3.2) \approx 9.760$ ,

(d)  $f'(2.0) \approx 0.1600$ ,  $f'(2.1) \approx -0.1000$ ,  $f'(2.2) \approx -0.3300$ ,  $f'(2.3) \approx -0.5500$ ,

17. For the endpoints of the tables, we use Formula (4.7). The other approximations come from Formula (4.6).

(a)  $f'(2.1) \approx 3.884$     $f'(2.2) \approx 2.896$     $f'(2.3) \approx 2.249$     $f'(2.4) \approx 1.836$     $f'(2.5) \approx 1.550$   
 $f'(2.6) \approx 1.348$

(b)  $f'(-3.0) \approx -5.883$     $f'(-2.8) \approx -5.467$     $f'(-2.6) \approx -5.059$     $f'(-2.4) \approx -4.650$   
 $f'(-2.2) \approx -4.208$     $f'(-2.0) \approx -3.875$

18. (a)

|       |            | $f'(0.4)$  |       |           | $f''(0.4)$ |
|-------|------------|------------|-------|-----------|------------|
| (4.1) | $h = 0.6$  | -0.8889958 | (4.8) | $h = 0.2$ | -1.191050  |
|       | $h = 0.4$  | -0.6979043 |       |           |            |
|       | $h = 0.2$  | -0.5486810 |       |           |            |
|       | $h = -0.2$ | -0.3104710 |       |           |            |
| (4.4) | $h = 0.2$  | -0.3994578 |       |           |            |
| (4.5) | $h = 0.2$  | -0.4295760 |       |           |            |

(b)

|       |            | $f'(0.4)$  |       |           | $f''(0.4)$ |
|-------|------------|------------|-------|-----------|------------|
| (4.1) | $h = 0.4$  | -1.059153  | (4.8) | $h = 0.4$ | -1.573943  |
|       | $h = 0.2$  | -0.8471275 |       | $h = 0.2$ | -1.492233  |
|       | $h = -0.2$ | -0.5486810 |       |           |            |
|       | $h = -0.4$ | -0.4295760 |       |           |            |
| (4.4) | $h = 0.2$  | -0.6351018 |       |           |            |
|       | $h = -0.2$ | -0.6677860 |       |           |            |
| (4.5) | $h = 0.4$  | -0.7443646 |       |           |            |
|       | $h = 0.2$  | -0.6979043 |       |           |            |
| (4.6) | $h = 0.2$  | -0.6824175 |       |           |            |

19. The approximation is  $-4.8 \times 10^{-9}$ .  $f''(0.5) = 0$ . The error bound is 0.35874. The method is very accurate since the function is symmetric about  $x = 0.5$ .
20. With  $h = 0.1$ , we have 36.641, and with  $h = 0.01$ , we have 36.5. The actual value is 36.5935.
21. (a)  $f'(0.2) \approx -0.1951027$       (b)  $f'(1.0) \approx -1.541415$       (c)  $f'(0.6) \approx -0.6824175$
22. We have the Taylor expansions:

$$\begin{aligned}
 f(x_0 - h) &= f(x_0) - hf'(x_0) + \frac{1}{2}h^2 f''(x_0) - \frac{1}{6}h^3 f'''(x_0) + \frac{1}{24}h^4 f^{(4)}(x_0) + O(h^5); \\
 f(x_0 + h) &= f(x_0) + hf'(x_0) + \frac{1}{2}h^2 f''(x_0) + \frac{1}{6}h^3 f'''(x_0) + \frac{1}{24}h^4 f^{(4)}(x_0) + O(h^5); \\
 f(x_0 + 2h) &= f(x_0) + 2hf'(x_0) + 2h^2 f''(x_0) + \frac{4}{3}h^3 f'''(x_0) + \frac{2}{3}h^4 f^{(4)}(x_0) + O(h^5); \\
 f(x_0 + 3h) &= f(x_0) + 3hf'(x_0) + \frac{9}{2}h^2 f''(x_0) + \frac{9}{2}h^3 f'''(x_0) + \frac{27}{8}h^4 f^{(4)}(x_0) + O(h^5).
 \end{aligned}$$

Thus,

$$\begin{aligned}
 Af(x_0 - h) + Bf(x_0 + h) + Cf(x_0 + 2h) + Df(x_0 + 3h) = \\
 f(x_0)(A + B + C + D) + f'(x_0)h[-A + B + 2C + 3D] + f''(x_0)h^2 \left( \frac{1}{2}A + \frac{1}{2}B + 2C + \frac{9}{2}D \right) \\
 + f'''(x_0)h^3 \left( -\frac{1}{6}A + \frac{1}{6}B + \frac{4}{3}C + \frac{9}{2}D \right) + f^{(4)}(x_0)h^4 \left( \frac{1}{24}A + \frac{1}{24}B + \frac{2}{3}C + \frac{27}{8}D \right).
 \end{aligned}$$

We want to eliminate the terms involving  $f''(x_0)$ ,  $f'''(x_0)$ , and  $f^{(4)}(x_0)$  and have the coefficient

of  $f'(x_0)$  equal 1. Thus,

$$\begin{aligned} -A + B + 2C + 3D &= 1 \\ \frac{1}{2}A + \frac{1}{2}B + 2C + \frac{9}{2}D &= 0 \\ -\frac{1}{6}A + \frac{1}{6}B + \frac{4}{3}C + \frac{9}{2}D &= 0 \\ \frac{1}{24}A + \frac{1}{24}B + \frac{2}{3}C + \frac{27}{8}D &= 0. \end{aligned}$$

The solution to this linear system is

$$A = -\frac{1}{4}, \quad B = \frac{3}{2}, \quad C = -\frac{1}{2}, \quad \text{and} \quad D = \frac{1}{12}.$$

Thus,

$$-\frac{1}{4}f(x_0-h) + \frac{3}{2}f(x_0+h) - \frac{1}{2}f(x_0+2h) + \frac{1}{12}f(x_0+3h) = f(x_0) \left( -\frac{1}{4} + \frac{3}{2} - \frac{1}{2} + \frac{1}{12} \right) + hf'(x_0) + O(h^5).$$

Solving for  $f'(x_0)$  gives

$$f'(x_0) = -\frac{1}{h} \left[ f(x_0) \frac{10}{12} + \frac{1}{4}f(x_0-h) - \frac{3}{2}f(x_0+h) + \frac{1}{2}f(x_0+2h) - \frac{1}{12}f(x_0+3h) \right] + O(h^4).$$

Finally,

$$f'(x_0) = \frac{1}{12h} [-3f(x_0-h) - 10f(x_0) + 18f(x_0+h) - 6f(x_0+2h) + f(x_0+3h)] + O(h^4).$$

23.  $f'(0.4) \approx -0.4249840$  and  $f'(0.8) \approx -1.032772$ .

24. (a) Assume that the computed values  $\tilde{f}(x_0+h)$  and  $\tilde{f}(x_0)$  are related to the true values  $f(x_0+h)$  and  $f(x_0)$  by the formulas  $f(x_0+h) = \tilde{f}(x_0+h) + e(x_0+h)$  and  $f(x_0) = \tilde{f}(x_0) + e(x_0)$ . The total error in the approximation becomes

$$f'(x_0) - \frac{\tilde{f}(x_0+h) - \tilde{f}(x_0)}{h} = \frac{e(x_0+h) - e(x_0)}{h} - \frac{h}{2}f''(\xi_0).$$

If  $|e(x_0+h)| < \varepsilon$ ,  $|e(x_0)| < \varepsilon$ , and  $|f''(\xi_0)| \leq M$ , then

$$\left| f'(x_0) - \frac{\tilde{f}(x_0+h) - \tilde{f}(x_0)}{h} \right| \leq \frac{2\varepsilon}{h} + \frac{hM}{2}.$$

(b) The function in Example 2 is

$$f(x) = xe^x, \quad \text{for } 1.8 \leq x \leq 2.2.$$

We have  $f'(x) = xe^x + e^x$  and  $f''(x) = xe^x + 2e^x$ . Thus,

$$M = \max_{1.8 \leq x \leq 2.2} |f''(x)| = f''(2.2) = 37.9050567.$$

The numbers in the table are given to 6 decimal places, so it is reasonable to let  $\varepsilon = 0.0000005$ . The optimal value of  $h$  is

$$h = 2\sqrt{\frac{\varepsilon}{M}} = 2\sqrt{\frac{0.0000005}{37.9050567}} = 0.000229703.$$

25. The three-point formulas give the results in the following table.

|       |    |      |      |      |      |      |
|-------|----|------|------|------|------|------|
| Time  | 0  | 3    | 5    | 8    | 10   | 13   |
| Speed | 79 | 82.4 | 74.2 | 76.8 | 69.4 | 71.2 |

26. The three-point formulas give the results in the following table.

|                  |       |       |       |       |       |
|------------------|-------|-------|-------|-------|-------|
| t                | 1.00  | 1.01  | 1.02  | 1.03  | 1.04  |
| $\varepsilon(t)$ | 2.400 | 2.403 | 3.386 | 5.352 | 7.320 |

27. The approximations eventually become zero since the numerator becomes zero.  
 28. By averaging the Taylor polynomials we have

$$f'''(x_0) = \frac{1}{h^3} \left[ -\frac{1}{2}f(x_0 - 2h) + f(x_0 - h) - f(x_0 + h) + \frac{1}{2}f(x_0 + 2h) \right] - \frac{h^2}{4}f^{(5)}(\xi),$$

where  $\xi$  is between  $x_0 - 2h$  and  $x_0 + 2h$ .

29. Since  $e'(h) = -\varepsilon/h^2 + hM/3$ , we have  $e'(h) = 0$  if and only if  $h = \sqrt[3]{3\varepsilon/M}$ . Also,  $e'(h) < 0$  if  $h < \sqrt[3]{3\varepsilon/M}$  and  $e'(h) > 0$  if  $h > \sqrt[3]{3\varepsilon/M}$ , so an absolute minimum for  $e(h)$  occurs at  $h = \sqrt[3]{3\varepsilon/M}$ .

## Exercise Set 4.2, page 184

1. (a)  $f'(1) \approx 1.0000109$  (b)  $f'(0) \approx 2.0000000$   
 (c)  $f'(1.05) \approx 2.2751459$  (d)  $f'(2.3) \approx -19.646799$
2. (a)  $f'(1) \approx 0.99999998$  (b)  $f'(0) \approx 1.9999999$   
 (c)  $f'(1.05) \approx 2.2751458$  (d)  $f'(2.3) \approx -19.646796$
3. (a)  $f'(1) \approx 1.001$  (b)  $f'(0) \approx 1.999$