(a) $x_n = 2a\left(\frac{1}{2}\right)^n$

(b) $$\begin{cases} x_1^e = \frac{1}{3} \\ x_2^e = \frac{1}{6} \\ x_n^e = \frac{5}{2} x_{n-1}^e - x_{n-2}^e, & n \geq 3 \end{cases}$$

solution : $x_n^e = \frac{2}{3}\left(\frac{1}{2}\right)^n$

Numerical solution :

$$\begin{cases} x_1^h = fl\left(\frac{1}{3}\right) = \frac{1}{3}(1+\delta_1) \\ x_2^h = fl\left(\frac{1}{6}\right) = \frac{1}{6}(1+\delta_2) \\ x_n^h = fl\left(fl\left(\frac{5}{2}\right) x_{n-1}^h - fl(1)\, x_{n-2}^h\right) \end{cases}$$

$$= \left(\frac{5}{2} x_{n-1}^h - x_{n-2}^h\right)(1+\delta_n)$$

$$= \frac{5}{2}(1+\delta_n)\, x_{n-1}^h - (1+\delta_n)\, x_{n-2}^h$$

rewrite $x_n^e = \frac{5}{2}(1+\delta_n)\, x_{n-1}^e - (1+\delta)\, x_{n-2}^e - \delta_n\left(\frac{5}{2} x_{n-1}^e - x_{n-2}^e\right)$

Equation for the error :

$$e_n = x_n^h - x_n^e$$

$$= (1+\delta_n)\left(\frac{5}{2} e_{n-1} - e_{n-2}\right) + \delta_n\left(x_n^e\right)$$

$$\Rightarrow \quad e_n = \tilde{e}_n + \tilde{\tilde{e}}_n$$

where $$\begin{cases} \tilde{e}_1 = \frac{1}{3}\delta_1 \\ \tilde{e}_2 = \frac{1}{6}\delta_2 \\ \tilde{e}_n = (1+\delta_n)\left(\frac{5}{2} e_{n-1} - e_{n-2}\right) \end{cases} \qquad \begin{cases} \tilde{\tilde{e}}_1 = 0 \\ \tilde{\tilde{e}}_2 = 0 \\ \tilde{\tilde{e}}_n = (1+\delta_n)\left(\frac{5}{2}\tilde{\tilde{e}}_{n-1} - \tilde{\tilde{e}}_{n-2}\right) + \frac{2\delta_n}{3}\left(\frac{1}{2}\right)^n \end{cases}$$

The solution $\tilde{\tilde{e}}_n$ can be approximated by assuming $\delta_n = 0$

$$\tilde{\tilde{e}}_n \cong d_1 (2)^n + d_2 \left(\frac{1}{2}\right)^n$$

$$\cong \frac{1}{18}(\delta_2 - \delta_1)(2)^n + \frac{4}{3}\left(\frac{2}{3}\delta_1 - \frac{1}{6}\delta_2\right)\left(\frac{1}{2}\right)^n$$

$\tilde{\tilde{e}}_n$ is more complicated, and grows at same rate with $\tilde{e}_n$

relative error

$$\frac{|e_n|}{x_n^e} \approx C \frac{2^n}{(2^{-n})} \simeq C 4^n$$

(c) if $a = 1$

$$\begin{cases} x_1^e = 1 \\ x_2^e = \frac{1}{2} \\ x_n^e = \frac{5}{2} x_n^e - x_{n-2}^e \quad , n \geq 3 \end{cases}$$

Solution : $x_n^e = 2\left(\frac{1}{2}\right)^n$

Numerical solution :

$$\begin{cases} x_1^h = fl(1) = 1 \\ x_1^h = fl\left(\frac{1}{2}\right) = \frac{1}{2} \\ x_n^h = fl\left(fl\left(\frac{5}{2}\right) x_{n-1}^h - fl(1)\, x_{n-2}^h\right) \end{cases}$$

$$= \left(\frac{5}{2} x_{n-1}^h - x_{n-2}^h\right)(1 + \delta_n)$$

the same as (b)

Equation for the error :

$$e_n = x_n^h - x_n^e$$

$\Rightarrow e_n = \frac{5}{2} e_{n-1} - e_{n-2} \ , n \geq 3$

and $\because e_1 = e_2 = 0$

$\Rightarrow e_n = 0$ #