1.(a) $f(x) = e^x - 0.99 \quad (x^* \approx 0)$

Let $g(x) = x + e^x - 0.99$

$$
\begin{aligned}
h(x) &= \alpha x + (1-\alpha) g(x) \\
&= \alpha x + (1-\alpha)(x + e^x - 0.99) \\
&= \alpha x + (1-\alpha) x + (1-\alpha)(e^x - 0.99) \\
&= x + (1-\alpha)(e^x - 0.99)
\end{aligned}
$$

Let $D = [-1, 1]$, Note that $0 \in D$.

choose $\alpha \ni h(D) \subset D$ and $|h'(x)| < 1$, for all $x \in D$

$\because h(D) \overset{\text{hope}}{\subset} D \quad \therefore \quad -1 < h(x) < 1 \quad \forall x \in [-1, 1]$

$\Rightarrow \quad -1 < x + (1-\alpha)(e^x - 0.99) < 1$

$$
\begin{cases}
\text{if } 0 \le x \le 1 \quad 0.01 < e^x - 0.99 < e - 0.99 \\
\Rightarrow 1 < \alpha < 1 + \frac{2}{e - 0.99} \approx 2.1572 \\
\text{if } -1 \le x < 0 \\
\Rightarrow 1 < \alpha < 1 + \frac{2}{0.99 - e^{-1}} \approx 4.2148
\end{cases}
$$

$\Rightarrow \quad 1 < \alpha < 2.1572$ —— ①

choose $\alpha \ni |h'(x)| < 1$ for all $x \in D$.

$\Rightarrow \quad -1 < 1 + (1-\alpha) e^x < 1$

$\Rightarrow \quad 1 < \alpha < 1 + \frac{2}{e^x} \Rightarrow 1 < \alpha < 1 + \frac{2}{e} \approx 1.7358.$ —— ②

by ①.② choose $1 < \alpha < 1 + \frac{2}{e}$, best α is $1 + \frac{1}{e} \approx 1.3679$ #

$x_0 = 1$

$x_1 = 0.364200646759728$

$x_2 = 0.198889241591437$

$x^* = -0.0100503358535 01$

$e^{x^*} - 0.99 \approx 0$

#

1.(b) $$P_n = P_{n-1} - \frac{f(P_{n-1})(P_{n-1} - P_{n-2})}{f(P_{n-1}) - f(P_{n-2})}$$

$x_0 = 2$, $x_1 = 1$, $x_2 = 0.629979586171579$

$x_3 = 0.239333565246312$

1.(c), $$P_n = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})}$$

$x_0 = 1$

$x_1 = 0.364200646759728$

$x_2 = 0.0520049100123368$

1.(d) $$P_n = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})} - \frac{f''(P_{n-1})}{2f'(P_{n-1})}\left[\frac{f(P_{n-1})}{f'(P_{n-1})}\right]^2$$

$$= P_{n-1} - \left(\frac{e^x - 0.99}{e^x}\right) - \frac{1}{2}\left(\frac{e^x - 0.99}{e^x}\right)^2$$

$x_0 = 0$

$x_1 = -0.0150$

$x_2 = -0.007519604031307$

$x_3 = -0.01132441201391$

$x^* = -0.010050335853501$, $f(x^*) \approx 0$

2.

(a) $F(x_1, x_2, x_3) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \sin(\sin x + 2y) - 0.01 \\ 5x + \sin(6y) - 0.02 \end{pmatrix}$

$$J(x_1, x_2, x_3) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\sin(x) + 2y)\cos x & \cos(\sin(x) + 2y) \cdot 2 \\ 5 & 6\cos(6y) \end{bmatrix}$$

$$\vec{X}_n = \vec{X}_{n-1} - \left(J(\vec{X}_{n-1})\right)^{-1} F(\vec{X}_{n-1})$$

$(x_0, y_0) = (0, 0)$

$(x_1, y_1) = (-0.005, 0.0075)$

$(x_2, y_2) = (-0.0049926257691816, 0.007496138580137469)$

final answer $= (-0.00499263687196 9, 0.00749639 1402366)$

(b)

Let $F(x,y) = \begin{pmatrix} \sin(\sin(x)+2y) - 0.01 \\ 5x + \sin(6y) - 0.02 \end{pmatrix}$

consider $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} F(x,y) = H(x,y)$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertiable matrix.

then if $\exists (x^*, y^*) \ni \begin{pmatrix} x^* \\ y^* \end{pmatrix} = H(x^*, y^*)$

$\Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} F(x^*, y^*) = 0 \Leftrightarrow F(x^*, y^*) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\because \begin{pmatrix} \frac{\partial H_1}{\partial x} & \frac{\partial H_1}{\partial y} \\ \frac{\partial H_2}{\partial x} & \frac{\partial H_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 + a\cos(\sin(x)+2y)\cdot\cos x + 5b & 2a\cos(\sin(x)+2y) + 6b\cos(6y) \\ c\cos(\sin(x)+2y)\cos x + 5d & 1 + 2c\cos(\sin(x)+2y) + 6d\cos(6y) \end{pmatrix}$

$\because (x^*, y^*) \approx (0,0)$

$\approx \begin{pmatrix} 1+a+5b & 2a+6b \\ c+5d & 1+2c+6d \end{pmatrix} \overset{\text{hope}}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow a = 3/2,\ b = -1/2,\ c = -\frac{5}{4},\ d = \frac{1}{4}$

so, Let $H = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3/2 & -1/2 \\ -5/4 & 1/4 \end{pmatrix} F(x,y)$,

we have $\left|\frac{\partial H_1}{\partial x}\right| \approx 0$, $\left|\frac{\partial H_1}{\partial y}\right| \approx 0$, $\left|\frac{\partial H_2}{\partial x}\right| \approx 0$, $\left|\frac{\partial H_2}{\partial y}\right| \approx 0$ when (x,y) near $(0,0)$

$(x_0, y_0) = (0,0)$

$(x_1, y_1) = (-0.005, 0.0075)$

$(x_2, y_2) = (-0.0049926257769182, 0.0074963855801375)$

$(x^*, y^*) = (-0.004992636871969, 0.0074963914602366)$

$F(x^*, y^*) \approx (0, 0)$ #

3, $u_i = -0.5\left(u_{i-1} + u_{i+1} \right) + \sin(i\pi/N)$

$= \left(b(i) - a_{i,i-1}\, u_{i-1} - a_{i,i+1}\, u_{i+1} \right) / a_{i,i}$

$\Rightarrow\ b(i) = \sin(i\pi/N),\quad A = (a_{ij}) = \begin{bmatrix} 1 & 0.5 & & \\ 0.5 & \ddots & \ddots & \\ & \ddots & \ddots & 0.5 \\ & & 0.5 & 1 \end{bmatrix}$

(a)

pseudo-code for:

Jacobi: $u_0 = 0$, $u_N = 0$, $u' = \text{zeros}(N-1, 1)$;

```
do k=1
   do i=1, ..., N-1
      u'_i = -0.5*(u_{i-1} + u_{i+1}) + sin(iπ/N)
   end i
   u_i = u'_i
end k
```

SOR:

$u_0 = 0$, $u_N = 0$,

```
do k=1
   do i=1, ..., N-1
      ugs_i = -0.5*(u_{i-1} + u_{i+1}) + sin(iπ/N)
      u_i = (1-w) u_i + w ugs_i
   end i
end k
```

(b). $A = \begin{pmatrix} 1 & 0.5 & & 0 \\ 0.5 & 1 & \ddots & \\ & \ddots & \ddots & 0.5 \\ 0 & & 0.5 & 1 \end{pmatrix}$, $\underline{K(A)} \approx 0.001655135$

By Theorem 7.27

$$\frac{\|x - \tilde{x}\|}{\|x\|} \leq K(A) \frac{\|r\|}{\|b\|}$$

where $\tilde{x}$ is an approximation to the solution of $Ax = b$, A is a nonsingular matrix, and r is the residual vector for $\tilde{x}$ ※

(c) pseudo-code for.

Jacobi :

$uo_{i,j} = un_{i,j} = 0$, for all $i=0$ or $i=N$ or $j=0$ or $j=N$

```
do k=1, ....,
    do j=1, ...., N-1,
        do i= 1, ...., N-1,
            un_ij = -0.25 *( uo_i-1,j + uo_i+1,j + uo_i,j-1 + uo_i,j+1)
                    + sin(iπ/N) sin(jπ/N)
        end i
    end j
    uo = un :
end k
```

SOR:

$u_{i,j} = 0$, for all $i=0$ or $i=N$ or $j=0$ or $j=N$

```
do k = 1, ....,
   do j = 1, ...., N-1
      do i = 1, ...., N-1
         u_GS = -0.25 * ( u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} ) + sin(iπ/N) sin(jπ/N)
         u_{i,j} = (1-w) u_{i,j} + w u_GS
      end i
   end j
end k
```

※

4. $p(x) = a_0 + a_1 x + a_2 x^2$

$$\min F(a_0, a_1, a_2) = \int_0^1 (e^x - p(x))^2 \, dx$$

$$\frac{\partial F}{\partial a_0} = \int_0^1 2(e^x - p(x)) \, dx = \left(2e^x - (2a_0 x + a_1 x^2 + \frac{2}{3} a_2 x^3)\right)\Big|_0^1$$

$$\frac{\partial F}{\partial a_1} = \int_0^1 2(e^x - p(x)) x \, dx = \left(2e^x(x-1) - (a_0 x^2 + \frac{2}{3} a_1 x^3 + \frac{1}{2} a_2 x^4)\right)\Big|_0^1$$

$$\frac{\partial F}{\partial a_2} = \int_0^1 2(e^x - p(x)) x^2 \, dx = \left(2e^x(x^2 - 2x + 2) - (\frac{2}{3} a_0 x^3 + \frac{1}{2} a_1 x^4 + \frac{2}{5} a_2 x^5)\right)\Big|_0^1$$

$$\Rightarrow \begin{cases} 2e - (2a_0 + a_1 + \frac{2}{3} a_2) - 2 = 0 \\ -(a_0 + \frac{1}{3} a_1 + \frac{1}{2} a_2) + 2 = 0 \\ 2e - (\frac{2}{3} a_0 + \frac{1}{2} a_1 + \frac{2}{5} a_2) - 4 = 0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} -2 & -1 & -\frac{2}{3} \\ -1 & -\frac{2}{3} & -\frac{1}{2} \\ -\frac{2}{3} & -\frac{1}{2} & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2-2e \\ -2 \\ 4-2e \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -9/2 & 18 & -15 \\ 18 & -96 & 90 \\ -15 & 90 & -90 \end{bmatrix} \begin{bmatrix} 2-2e \\ -2 \\ 4-2e \end{bmatrix} \approx \begin{bmatrix} 1.0130 \\ 0.8511 \\ 0.8392 \end{bmatrix}$$

5.

$$\frac{p_{n+1}-p}{p_n-p} \approx \frac{p_{n+2}-p}{p_{n+1}-p}$$

$$\Rightarrow (p_{n+1}-p)^2 \approx (p_{n+2}-p)(p_n-p)$$

$$\Rightarrow p_{n+1}^2 - 2p_{n+1}p + p^2 \approx p_{n+2}p_n - (p_n+p_{n+2})p + p^2$$

$$\Rightarrow (p_{n+2}+p_n-2p_{n+1})p \approx p_{n+2}p_n - p_{n+1}^2$$

$$\Rightarrow p \approx \frac{p_{n+2}p_n - p_{n+1}^2}{p_{n+2}-2p_{n+1}+p_n}$$

$$= \frac{p_np_{n+2} - 2p_np_{n+1} + p_n^2 - p_{n+1}^2 + 2p_np_{n+1} - p_n^2}{p_{n+2}-2p_{n+1}+p_n}$$

$$= p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}$$

$$a_n = \left(\frac{1}{2}\right)^2$$

$$\hat{a}_n = \hat{a}_n - \frac{(\Delta a_n)^2}{\Delta^2 a_n} = a_n - \frac{(a_{n+1}-a_n)^2}{a_{n+2}-2a_{n+1}+a_n}$$

$$= \left(\frac{1}{2}\right)^n - \frac{\left(\left(\frac{1}{2}\right)^{n+1} - \left(\frac{1}{2}\right)^n\right)^2}{\left(\frac{1}{2}\right)^{n+2} - 2\left(\frac{1}{2}\right)^{n+1} + \left(\frac{1}{2}\right)^n}$$

$$= \left(\frac{1}{2}\right)^n - (2)^{n+2}\left(\left(\frac{1}{2}\right)^{2n+2} - 2\left(\frac{1}{2}\right)^{n+1}\left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{2n}\right)$$

$$= \left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^n = 0$$