Chapter 4

Iterative Methods for Solving Systems of Linear Equations

In this chapter we present some iterative methods for solving a linear system of equations $Ax = b$.

4.1 Classic Iterative Methods

4.1.1 Basic Concept

First of all we give an example to illustrate the process of iterative methods for solving systems of linear equations.

Consider solving

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}.$$

This system has the exact solution $x_1 = x_2 = 1$. Equivalently we can write the system as

$$\begin{cases} 3x_1 + 2x_2 = 5 \\ x_1 + 4x_2 = 5 \end{cases}$$

This implies that

$$\begin{cases} x_1 = \frac{1}{3}(5 - 2x_2) \\ x_2 = \frac{1}{4}(5 - x_1) \end{cases}$$

A naieve idea is to solve the system by

$$\begin{cases} x_1^{(k)} = \frac{1}{3}(5 - 2x_2^{(k-1)}) \\ x_2^{(k)} = \frac{1}{4}(5 - x_1^{(k-1)}) \end{cases}$$

that is, to use the iterative formulation

$$\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \left(\begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \end{bmatrix} \right)$$

If we choose the initial guess $x_1^{(0)} = x_2^{(0)} = 0$, we would otain

$$\begin{aligned} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix} &= \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \left(\begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1.6667 \\ 1.2500 \end{bmatrix} \\ \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix} &= \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \left(\begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1.6667 \\ 1.2500 \end{bmatrix} \right) = \begin{bmatrix} 0.8333 \\ 0.8333 \end{bmatrix} \end{aligned}$$

By repeating the process, we have the following table

k	3	4	5	6	7
$x_1^{(k)}$	1.1111	0.9722	1.0185	0.9954	1.0031
$x_2^{(k)}$	1.0417	0.9722	1.0000	0.9954	1.0012

From this example, we observe that the basic idea is to split the coefficient matrix A into

$$A = M - (M - A), \tag{4.1}$$

for some matrix M, which is called the splitting matrix. Here we assume that A and M are both nonsingular. Then the original problem is rewritten in the equivalent form

$$Mx = (M - A)x + b.$$

This suggests an iterative process

$$x^{(k)} = (I - M^{-1}A)x^{(k-1)} + M^{-1}b \equiv Tx^{(k-1)} + c, \tag{4.2}$$

where T is usually called the iteration matrix. The initial vector $x^{(0)}$ can be arbitrary or be chosen according to certain conditions.

Two criteria for choosing the splitting matrix M are

1. $x^{(k)}$ is easily computed. More precisely, the system $Mx^{(k)} = y$ is easy to solve;

2. the sequence $\{x^{(k)}\}$ converges rapidly to the exact solution.

Note that one way to achieve the second goal is to choose M so that M^{-1} approximate A^{-1},

In the following subsections, we will introduce some of the mostly commonly used classic iterative methods.

4.1.2 Richard's Method

When we choose $M = I$ such that $A = I - (I - A)$, we obtain the iteration procedure

$$x^{(k)} = (I - A)x^{(k-1)} + b = x^{(k-1)} - Ax^{(k-1)} + b \equiv x^{(k-1)} + r^{(k-1)}, \tag{4.3}$$

This algorithm is called the Richard's method.

Algorithm 4.1 (Richard's Method)

for $k = 1, 2, \ldots$ **do**
  for $i = 1, 2, \ldots, n$ **do**
    $r_i^{(k-1)} = b_i - \sum_{j=1}^{n} a_{ij} x_j^{(k-1)}$
    $x_i^{(k)} = x_i^{(k-1)} + r_i^{(k-1)}$
  end for
end for

Two n-vectors are required to implement this algorithm.

4.1.3 Jacobi Method

If we decompose the coefficient matrix A as

$$A = L + D + U, \tag{4.4}$$

where D is the diagonal part, L is the strictly lower triangular part, and U is the strictly upper triangular part, of A, and choose $M = D$, then we derive the iterative formulation for Jacobi method:

$$x^{(k)} = -D^{-1}(L + U)x^{(k-1)} + D^{-1}b. \tag{4.5}$$

With this method, the iteration matrix $T = -D^{-1}(L + U)$ and $c = D^{-1}b$. Each component $x_i^{(k)}$ can be computed by

$$x_i^{(k)} = \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k-1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \right) \Bigg/ a_{ii}. \tag{4.6}$$

Algorithm 4.2 (Jacobi Method)

for $k = 1, 2, \ldots$ **do**
  for $i = 1, 2, \ldots, n$ **do**
    $x_i^{(k)} = \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k-1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \right) \Big/ a_{ii}$
  end for
end for

Two n-vectors are required to implement this algorithm. A big advantage of Jacobi method is that $x_i^{(k)}, i = 1, \dots, n$, can be computed in parallel at each iteration k.

Another important observation is that in Jacobi method, only the components of $x^{(k-1)}$ are used to compute $x^{(k)}$. When computing $x_i^{(k)}$ for $i > 1$, $x_1^{(k)}, \dots, x_{i-1}^{(k)}$ have already been computed and are likely to be better approximations to the exact $x_1, \dots, x_{i-1}$ than $x_1^{(k-1)}, \dots, x_{i-1}^{(k-1)}$. It seems reasonable to compute $x_i^{(k)}$ using these most recently computed values. This improvement induce the Gauss-Seidel method which is to be discussed in the next subsection.

4.1.4 Gauss-Seidel Method

The Gauss-Seidel method sets $M = D + L$ and defines the iteration as

$$x^{(k)} = -(D+L)^{-1}Ux^{(k-1)} + (D+L)^{-1}b. \tag{4.7}$$

That is, Gauss-Seidel method uses $T = -(D+L)^{-1}U$ as the iteration matrix. The formulation above can be rewritten as

$$x^{(k)} = -D^{-1}\left(Lx^{(k)} + Ux^{(k-1)} - b\right). \tag{4.8}$$

Hence each component $x_i^{(k)}$ can be computed by

$$x_i^{(k)} = \left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}\right) \Big/ a_{ii}. \tag{4.9}$$

Algorithm 4.3 (Gauss-Seidel Method)

for $k = 1, 2, \dots$ **do**
  for $i = 1, 2, \dots, n$ **do**
    $x_i^{(k)} = \left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}\right) \Big/ a_{ii}$
  end for
end for

At each iteration, since $x_i^{(k)}$ can not be computed until $x_1^{(k)}, \dots, x_{i-1}^{(k)}$ are available, Gauss-Seidel method is not a parallel algorithm in nature. Moreover, only one n-vector is required for implementation in theory, but two are usually used in practice.

4.1.5 Successive Over Relaxation (SOR) Method

The successive over relaxation (SOR) method choose $M = \omega^{-1}(D + \omega L)$, where $0 < \omega < 2$ is called the relaxation parameter, and defines the iteration

$$(D + \omega L)x^{(k)} = \left[(1-\omega)D - \omega U\right]x^{(k-1)} + \omega b. \tag{4.10}$$

Hence the iteration matrix $T = (D + \omega L)^{-1}((1 - \omega)D - \omega U)$. Each component $x_i^{(k)}$ can be computed by the formulation

$$x_i^{(k)} = \omega \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \right) \Big/ a_{ii} + (1 - \omega) x_i^{(k-1)}. \tag{4.11}$$

The question of choosing a good relaxation parameter ω is a very complex topic.

4.1.6 Symmetric Successive Over Relaxation (SSOR) Method

In theory the symmetric successive over relaxation (SSOR) method chooses the splitting matrix $M = \frac{1}{\omega(2-\omega)}(D + \omega L)D^{-1}(D + \omega U)$ and iterates with the iteration matrix

$$T = (D + \omega U)^{-1} ((1 - \omega)D - \omega L) (D + \omega L)^{-1} ((1 - \omega)D - \omega U). \tag{4.12}$$

The idea is in fact to implement the SOR formulation twice, one forward and one backward, at each iteration. That is, SSOR method defines

$$(D + \omega L)x^{(k-\frac{1}{2})} = ((1 - \omega)D - \omega U) x^{(k-1)} + \omega b \tag{4.13}$$

$$(D + \omega U)x^{(k)} = ((1 - \omega)D - \omega L) x^{(k-\frac{1}{2})} + \omega b \tag{4.14}$$

Each component $x_i^{(k)}$ is obtained by first computing

$$x_i^{(k-\frac{1}{2})} = \omega \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k-\frac{1}{2})} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \right) \Big/ a_{ii} + (1 - \omega) x_i^{(k)} \tag{4.15}$$

followed by

$$x_i^{(k)} = \omega \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-\frac{1}{2})} \right) \Big/ a_{ii} + (1 - \omega) x_i^{(k-\frac{1}{2})}. \tag{4.16}$$

4.2 Convergence Analysis

Definition 4.1 (Eigenvalue) *Suppose A is a square matrix. The number λ is called an eigenvalue of A if there exists a nonzero vector x such that $Ax = \lambda x$. And x is called the corresponding eigenvector of A.*

Definition 4.2 *The polynomial defined by*

$$p(\lambda) = \det(\lambda I - A)$$

is called the characteristic polynomial of A.

Theorem 4.1 *λ is an eigenvalue of A if and only if λ is a root of the characteristic polynomial of A, that is, $p(\lambda) = 0$.*

Definition 4.3 (Spectrum and Spectral Radius) *The set of all eigenvalues of a matrix A is called the spectrum of A and is denoted by $\lambda(A)$. The spectral radius of A is*

$$\rho(\lambda) = \max\{|\lambda| | \lambda \in \lambda(A)\}. \tag{4.17}$$

Lemma 4.1 *If $A \in \mathbb{R}^{n\times n}$, then*

1. $\|A\|_2 = \sqrt{\rho(A^T A)}$;
2. $\rho(A) \le \|A\|$ *for any subordinate matrix norm.*

Proof: Proof for the second part. Suppose λ is an eigenvalue of A and $x \neq 0$ is a corresponding eigenvector such that $Ax = \lambda x$ and $\|x\| = 1$. Then

$$|\lambda| = |\lambda|\|x\| = \|\lambda x\| = \|Ax\| \le \|A\|\|x\| = \|A\|,$$

that is, $|\lambda| \le \|A\|$. Since λ is arbitrary, this implies that $\rho(A) = \max|\lambda| \le \|A\|$. ∎

Theorem 4.2 *For any A and any $\epsilon > 0$, there exists a subordinate norm such that*

$$\rho(A) < \|A\| < \rho(A) + \epsilon. \tag{4.18}$$

Lemma 4.2 *If $\rho(A) < 1$, then $(I - A)^{-1}$ exists and $(I - A)^{-1} = \sum_{i=0}^{\infty} A^i = I + A + A^2 + \cdots$.*

Proof: Suppose that λ is an eigenvalue of A, then $1 - \lambda$ is an eigenvalue of $I - A$. Since $|\lambda| \le \rho(A) < 1$, this implies $1 - \lambda \neq 0$. Hence 0 is not an eigenvalue of $I - A$, which means $(I - A)$ is nonsingular.

Next we show that $(I - A)^{-1} = I + A + A^2 + \cdots$. Since

$$(I - A)\left(\sum_{i=0}^{m} A^i\right) = I - A^{m+1},$$

and $\rho(A) < 1$ implies $\|A^m\| \to 0$ as $m \to \infty$, we have

$$(I - A)\left(\lim_{m\to\infty} \sum_{i=0}^{m} A^i\right) = I.$$

This proves $(I - A)^{-1} = \sum_{k=1}^{\infty} A^k$.

Lemma 4.3 *Suppose that $A \in \mathbb{R}^{n\times n}$ and $\|\cdot\|$ is a subordinate matrix norm. If $\|A\| < 1$, then $I - A$ is nonsingular and*

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k, \tag{4.19}$$

with

$$\|(I - A)^{-1}\| \le \frac{1}{1 - \|A\|}. \tag{4.20}$$

Proof: Suppose $I - A$ is singular. Then there exist $x \in \mathbb{R}^n$, $x \neq 0$ (so $\|x\| \neq 0$) such that $(I - A)x = 0$. Thus $x = Ax$ and $\|x\| = \|Ax\| \leq \|A\|\|x\|$, which gives $\|A\| > 1$. However, this contradicts to the assumption that $\|A\| < 1$. Hence $I - A$ is nonsingular.

Next, one can verify that

$$(I - A)\left(\sum_{k=0}^{m} A^k\right) = I - A^{m+1}.$$

Since $\|A\| < 1$, $\lim\limits_{m\to\infty} A^m = 0$, hence

$$(I - A)\left(\sum_{k=0}^{\infty} A^k\right) = (I - A)\left(\lim_{m\to\infty}\sum_{k=0}^{m} A^k\right) = I - \lim_{m\to\infty} A^{m+1} = I.$$

This shows that $(I - A)^{-1} = \sum\limits_{k=0}^{\infty} A^k$. Finally, since $\|A\| < 1$,

$$\|(I - A)^{-1}\| = \left\|\sum_{k=0}^{\infty} A^k\right\| \leq \sum_{k=0}^{\infty} \|A^k\| \leq \sum_{k=0}^{\infty} \|A\|^k = \frac{1}{1 - \|A\|}.$$

■

Theorem 4.3 *The following statements are equivalent.*

1. *A is a convergent matrix, i.e.,* $A^k \to \infty$ *as* $k \to \infty$*;*
2. $\lim\limits_{k\to\infty} \|A^k\| = 0$ *for some subordinate matrix norm;*
3. $\lim\limits_{k\to\infty} \|A^k\| = 0$ *for all subordinate matrix norm;*
4. $\rho(A) < 1$*;*
5. $\lim\limits_{k\to\infty} A^k x = 0$ *for any* x*.*

Theorem 4.4 *For any* $x^{(0)} \in \mathbb{R}^n$*, the sequence produced by*

$$x^{(k)} = Tx^{(k-1)} + c, \quad k = 1, 2, \dots, \tag{4.21}$$

converges to the unique solution of $x = Tx + c$ *if and only if*

$$\rho(T) < 1. \tag{4.22}$$

Proof: Suppose $\rho(T) < 1$. The sequence of vectors $x^{(k)}$ produced by the iterative formulation are

$$\begin{aligned}
x^{(1)} &= Tx^{(0)} + c \\
x^{(2)} &= Tx^{(1)} + c = T^2 x^{(0)} + (T + I)c \\
x^{(3)} &= Tx^{(2)} + c = T^3 x^{(0)} + (T^2 + T + I)c \\
&\vdots
\end{aligned}$$

In general

$$x^{(k)} = T^k x^{(0)} + (T^{k-1} + T^{k-2} + \cdots T + I)c.$$

Since $\rho(T) < 1$, $\lim_{k=0}^{\infty} T^k x^{(0)} = 0$ for any $x^{(0)} \in \mathbb{R}^n$. Then

$$x^{(k)} \to (T^{k-1} + T^{k-2} + \cdots T + I)c, \quad \text{as} \quad k \to \infty.$$

By Lemma

$$x^{(k)} \to (I-T)^{-1}c \quad \text{as} \quad k \to \infty.$$

Conversely, suppose that the sequence of vectors $x^{(k)}$ converges to $x = (I-T)^{-1}c$. Since

$$x - x^{(k)} = Tx + c - Tx^{(k-1)} - c = T(x - x^{(k-1)}) = T^2(x - x^{(k-2)}) = \cdots = T^k(x - x^{(0)}).$$

Let $z = x - x^{(0)}$. Then

$$\lim_{k\to\infty} T^k z \lim_{k\to\infty} (x - x^{(k)}) = 0.$$

It follows from theorem $\rho(T) < 1$.

Corollary 4.1 *If $\|T\| < 1$ for some subordinate matrix norm, then the sequence produced by*

$$x^{(k)} = Tx^{(k-1)} + c$$

converges to the solution of $Ax = b$ for any initial vector $x^{(0)}$.

Proof: Since $\rho(T) < \|T\|$ for any subordinate matrix norm, the result follows immediately from the previous theorem.

Theorem 4.5 *If $\delta = \|T\| < 1$, then*

$$\|x^{(k)} - x\| \leq \frac{\delta}{1-\delta}\|x^{(k)} - x^{(k-1)}\|. \tag{4.23}$$

Proof: Since $x^{(k)} - x = T(x^{(k-1)} - x)$,

$$\|x^{(k)} - x\| \leq \|T\|\|x^{(k-1)} - x\| = \delta\|x^{(k-1)} - x^{(k)} + x^{(k)} - x\| \leq \delta\|x^{(k-1)} - x^{(k)}\| + \delta\|x^{(k)} - x\|,$$

and $1 - \delta > 0$, we obtain

$$\|x^{(k)} - x\| \leq \frac{\delta}{1-\delta}\|x^{(k)} - x^{(k-1)}\|.$$

■

This theorem implies that we can stop the iteration if $\|x^{(k)} - x^{(k-1)}$ is less than a small tolerance.

Theorem 4.6 *If $\|T\| < 1$, then the sequence $x^{(k)}$ converges to x for any initial $x^{(0)}$ and*

1. $\|x - x^{(k)}\| \leq \|T\|^k \|x - x^{(0)}\|$
2. $\|x - x^{(k)}\| \leq \frac{\|T\|^k}{1-\|T\|} \|x^{(1)} - x^{(0)}\|$.

Proof: Since $x = Tx + c$ and $x^{(k)} = Tx^{(k-1)} + c$,

$$\begin{aligned} x - x^{(k)} &= Tx + c - Tx^{(k-1)} - c \\ &= T(x - x^{(k-1)}) \\ &= T^2(x - x^{(k-2)}) = \cdots\cdots = T^k(x - x^{(0)}). \end{aligned}$$

The first statement can then be derived

$$\|x - x^{(k)}\| = \|T^k(x - x^{(0)})\| \leq \|T\|^k \|x - x^{(0)}\|.$$

For the second result, we first show that $\|x^{(n)} - x^{(n-1)}\| \leq \|T\|^{n-1} \|x^{(1)} - x^{(0)}\|$ for any $n \geq 1$. Since

$$\begin{aligned} x^{(n)} - x^{(n-1)} &= Tx^{(n-1)} + c - Tx^{(n-2)} - c \\ &= T(x^{(n-1)} - x^{(n-2)}) \\ &= T^2(x^{(n-2)} - x^{(n-3)}) = \cdots\cdots = T^{n-1}(x^{(1)} - x^{(0)}), \end{aligned}$$

$$\|x^{(n)} - x^{(n-1)}\| \leq \|T\|^{n-1} \|x^{(1)} - x^{(0)}\|.$$

Let $m \geq k$,

$$\begin{aligned} x^{(m)} - x^{(k)} &= \left(x^{(m)} - x^{(m-1)}\right) + \left(x^{(m-1)} - x^{(m-2)}\right) + \cdots + \left(x^{(k+1)} - x^{(k)}\right) \\ &= T^{m-1}\left(x^{(1)} - x^{(0)}\right) + T^{m-2}\left(x^{(1)} - x^{(0)}\right) + \cdots + T^k\left(x^{(1)} - x^{(0)}\right) \\ &= \left(T^{m-1} + T^{m-2} + \cdots T^k\right)\left(x^{(1)} - x^{(0)}\right), \end{aligned}$$

hence

$$\begin{aligned} \|x^{(m)} - x^{(k)}\| &\leq \left(\|T\|^{m-1} + \|T\|^{m-2} + \cdots + \|T\|^k\right) \|x^{(1)} - x^{(0)}\| \\ &= \|T\|^k \left(\|T\|^{m-k-1} + \|T\|^{m-k-2} + \cdots + 1\right) \|x^{(1)} - x^{(0)}\|. \end{aligned}$$

Since $\lim_{m\to\infty} x^{(m)} = x$,

$$\begin{aligned} \|x - x^{(k)}\| &= \lim_{m\to\infty} \|x^{(m)} - x^{(k)}\| \\ &\leq \lim_{m\to\infty} \|T\|^k \left(\|T\|^{m-k-1} + \|T\|^{m-k-2} + \cdots + 1\right) \|x^{(1)} - x^{(0)}\| \\ &= \|T\|^k \|x^{(1)} - x^{(0)}\| \lim_{m\to\infty} \left(\|T\|^{m-k-1} + \|T\|^{m-k-2} + \cdots + 1\right) \\ &= \|T\|^k \frac{1}{1 - \|T\|} \|x^{(1)} - x^{(0)}\|. \end{aligned}$$

This proves the second result. ■

Theorem 4.7 *If A is strictly diagonal domainant, then both the Jacobi and Gauss-Seidel methods converges for any initial vector $x^{(0)}$.*

Proof: By assumption, A is strictly diagonal dominant, hence $a_{ii} \neq 0$ (otherwise A is singular) and

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad i = 1, 2, \ldots, n.$$

For Jacobi method, the iteration matrix $T_J = -D^{-1}(L+U)$ has entries

$$[T_J]_{ij} = \begin{cases} -\frac{a_{ij}}{a_{ii}} & i \neq j \\ 0 & i = j \end{cases}$$

Hence

$$\|T_J\|_\infty = \max_{1\leq i\leq n} \left| \sum_{j=1, j\neq i}^{n} \frac{a_{ij}}{a_{ii}} \right| = \max_{1\leq i\leq n} \frac{1}{|a_{ii}|} \sum_{j=1, j\neq i}^{n} |a_{ij}| < 1,$$

and this implies that the Jacobi method converges.

For Gauss-Seidel method, the iteration matrix $T_{GS} = -(D+L)^{-1}U$. Let λ be any eigenvalue of T_{GS} and y, $\|y\|_\infty = 1$, is a corresponding eigenvector. Thus

$$T_{GS}y = \lambda y \quad \Longrightarrow \quad -Uy = \lambda(D+L)y.$$

Hence for $i = 1, \ldots, n$,

$$-\sum_{j=i+1}^{n} a_{ij}y_j = \lambda \sum_{j=1}^{i} a_{ij}y_j = \lambda a_{ii}y_i + \sum_{j=1}^{i-1} a_{ij}y_j.$$

This gives

$$\lambda a_{ii}y_i = -\lambda \sum_{j=1}^{i-1} a_{ij}y_j + \sum_{j=i+1}^{n} a_{ij}y_j$$

and

$$|\lambda||a_{ii}||y_i| \leq |\lambda| \sum_{j=1}^{i-1} |a_{ij}||y_j| + \sum_{j=i+1}^{n} |a_{ij}||y_j|.$$

Choose the index k such that $|y_k| = 1 \geq |y_j|$ (this index can always be found since $\|y\|_\infty = 1$). Then

$$|\lambda||a_{kk}| \leq |\lambda| \sum_{j=1}^{k-1} |a_{kj}| + \sum_{j=k+1}^{n} |a_{kj}|$$

which gives

$$|\lambda| \le \frac{\sum_{j=k+1}^{n} |a_{kj}|}{|a_{kk}| - \sum_{j=1}^{k-1} |a_{kj}|} < \frac{\sum_{j=k+1}^{n} |a_{kj}|}{\sum_{j=k+1}^{n} |a_{kj}|} = 1$$

Since λ is arbitrary, $\rho(T_{GS}) < 1$. This means the Gauss-Seidel method converges. ■

Theorem 4.8 *If A is positive definite and the relaxation parameter ω satisfying $0 < \omega < 2$, then the SOR iteration converges for any initial vector $x^{(0)}$.*

Theorem 4.9 *If A is positive definite and tridiagonal, then $\rho(T_{GS}) = [\rho(T_J)]^2 < 1$ and the optimal choice of ω for the SOR iteration is*

$$\omega = \frac{2}{1 + \sqrt{1 - [\rho(T_J)]^2}}. \tag{4.24}$$

With this choice of ω, $\rho(T_{SOR}) = \omega - 1$.