Numerical Analysis II Numerical solutions of nonlinear systems of equations

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Spring 2011



- Fixed points for functions of several variables
- 2 Newton's method
- Quasi-Newton methods
- 4 Steepest Descent Techniques



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Theorem

Let $f:D\subset\mathbb{R}^n\to\mathbb{R}$ be a function and $x_0\in D$. If all the partial derivatives of f exist and \exists $\delta>0$ and $\alpha>0$ such that \forall $\|x-x_0\|<\delta$ and $x\in D$, we have

$$\left. \frac{\partial f(x)}{\partial x_j} \right| \le \alpha, \ \forall \ j = 1, 2, \dots, n,$$

then f is continuous at x_0 .

Definition (Fixed Point)

A function G from $D \subset \mathbb{R}^n$ into \mathbb{R}^n has a fixed point at $p \in D$ if G(p) = n.



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Let $D = \{(x_1, \dots, x_n)^T; a_i \leq x_i \leq b_i, \forall i = 1, \dots, n\} \subset \mathbb{R}^n$. Suppose $G: D \to \mathbb{R}^n$ is a continuous function with $G(x) \in D$ whenever $x \in D$. Then G has a fixed point in D.

Suppose, in addition, G has continuous partial derivatives and a constant $\alpha < 1$ exists with

$$\left| \frac{\partial g_i(x)}{\partial x_j} \right| \le \frac{\alpha}{n}, \text{ whenever } x \in D,$$

for $j=1,\ldots,n$ and $i=1,\ldots,n$. Then, for any $x^{(0)}\in D$,

$$x^{(k)} = G(x^{(k-1)}), \quad ext{for each } k \ge 1$$

converges to the unique fixed point $p \in D$ and

$$\|x^{(k)} - p\|_{\infty} \le \frac{\alpha^k}{1 - \alpha} \|x^{(1)} - x^{(0)}\|_{\infty}$$

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Example

Consider the nonlinear system

$$3x_1 - \cos(x_2 x_3) - \frac{1}{2} = 0,$$

$$x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 = 0,$$

$$e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0.$$

• Fixed-point problem:

Change the system into the fixed-point problem:

$$x_1 = \frac{1}{3}\cos(x_2x_3) + \frac{1}{6} \equiv g_1(x_1, x_2, x_3),$$

$$x_2 = \frac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1 \equiv g_2(x_1, x_2, x_3),$$

$$x_3 = -\frac{1}{20}e^{-x_1x_2} - \frac{10\pi - 3}{60} \equiv g_3(x_1, x_2, x_3).$$

Let $G: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $G(x) = [g_1(x), g_2(x), g_3(x)]^T$.



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Let $G: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $G(x) = [g_1(x), g_2(x), g_3(x)]^T$.



- G has a unique point in $D \equiv [-1,1] \times [-1,1] \times [-1,1]$:
 - ightharpoonup Existence: $\forall x \in D$,

$$\begin{split} |g_1(x)| &\leq \frac{1}{3} |\cos(x_2 x_3)| + \frac{1}{6} \leq 0.5, \\ |g_2(x)| &= \left| \frac{1}{9} \sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1 \right| \leq \frac{1}{9} \sqrt{1 + \sin 1 + 1.06} - 0.1 < 0.09, \\ |g_3(x)| &= \frac{1}{20} e^{-x_1 x_2} + \frac{10\pi - 3}{60} \leq \frac{1}{20} e^{+\frac{10\pi - 3}{60}} < 0.61, \end{split}$$

it implies that $G(x) \in D$ whenever $x \in D$

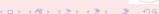
Uniqueness:

$$\left| \frac{\partial g_1}{\partial x_1} \right| = 0, \ \left| \frac{\partial g_2}{\partial x_2} \right| = 0 \ \text{and} \ \left| \frac{\partial g_3}{\partial x_3} \right| = 0,$$

as well as

$$\left|\frac{\partial g_1}{\partial x_2}\right| \leq \frac{1}{3}|x_3| \cdot |\sin(x_2 x_3)| \leq \frac{1}{3}\sin 1 < 0.281$$





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$$\begin{vmatrix} \frac{\partial g_2}{\partial x_1} \end{vmatrix} = \frac{|x_1|}{9\sqrt{x_1^2 + \sin x_3 + 1.06}} < \frac{1}{9\sqrt{0.218}} < 0.238,$$

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$$\left| \frac{\partial g_i}{\partial x_j} \right| \le 0.281, \ \forall \ i, j.$$

Similarly, $\partial g_i/\partial x_j$ are continuous on D for all i and j. Consequently, has a unique fixed point in D.

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Similarly, $\partial g_i/\partial x_j$ are continuous on D for all i and j. Consequently, G has a unique fixed point in D.

Approximated solution:

Fixed-point iteration (I): Choosing $x^{(0)} = [0.1, 0.1, -0.1]^T$, the sequence $\{x^{(k)}\}$ is generated by

$$\begin{array}{rcl} x_1^{(k)} & = & \frac{1}{3}\cos x_2^{(k-1)}x_3^{(k-1)} + \frac{1}{6}, \\ \\ x_2^{(k)} & = & \frac{1}{9}\sqrt{\left(x_1^{(k-1)}\right)^2 + \sin x_3^{(k-1)} + 1.06} - 0.1, \\ \\ x_3^{(k)} & = & -\frac{1}{20}e^{-x_1^{(k-1)}x_2^{(k-1)}} - \frac{10\pi - 3}{60}. \end{array}$$

Result

		1.2×10^{-5}
		3.1×10^{-7}

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Result:

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k	$x_1^{(k)}$	$x_2^{(k)}$	$x_{3}^{(k)}$	$ x^{(k)} - x^{(k-1)} _{\infty}$
0	0.10000000	0.10000000	-0.10000000	
1	0.49998333	0.00944115	-0.52310127	0.423
2	0.49999593	0.00002557	-0.52336331	$9.4 imes 10^{-3}$
3	0.50000000	0.00001234	-0.52359814	2.3×10^{-4}
4	0.50000000	0.00000003	-0.52359847	1.2×10^{-5}
5	0.50000000	0.00000002	-0.52359877	3.1×10^{-7}

- Approximated solution (cont.):
 - ► Accelerate convergence of the fixed-point iteration:

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as in the Gauss-Seidel method for linear systems.

► Result

		3.8×10^{-8}
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2	0.49997747	0.00002815	-0.52359807	2.2×10^{-2}
3	0.50000000	0.00000004	-0.52359877	2.8×10^{-5}
4	0.50000000	0.00000000	-0.52359877	3.8×10^{-8}
				7.

First consider solving the following system of nonlinear equations:

$$\begin{cases} f_1(x_1, x_2) = 0, \\ f_2(x_1, x_2) = 0. \end{cases}$$

Suppose $(x_1^{(k)},x_2^{(k)})$ is an approximation to the solution of the system above, and we try to compute $h_1^{(k)}$ and $h_2^{(k)}$ such that $(x_1^{(k)}+h_1^{(k)},x_2^{(k)}+h_2^{(k)})$ satisfies the system. By the Taylor's theorem fo two variables,

$$0 = f_{1}(x_{1}^{(k)} + h_{1}^{(k)}, x_{2}^{(k)} + h_{2}^{(k)})$$

$$\approx f_{1}(x_{1}^{(k)}, x_{2}^{(k)}) + h_{1}^{(k)} \frac{\partial f_{1}}{\partial x_{1}}(x_{1}^{(k)}, x_{2}^{(k)}) + h_{2}^{(k)} \frac{\partial f_{1}}{\partial x_{2}}(x_{1}^{(k)}, x_{2}^{(k)})$$

$$0 = f_{2}(x_{1}^{(k)} + h_{1}^{(k)}, x_{2}^{(k)} + h_{2}^{(k)})$$

$$\approx f_{2}(x_{1}^{(k)}, x_{2}^{(k)}) + h_{1}^{(k)} \frac{\partial f_{2}}{\partial x_{1}}(x_{1}^{(k)}, x_{2}^{(k)}) + h_{2}^{(k)} \frac{\partial f_{2}}{\partial x_{2}}(x_{1}^{(k)}, x_{2}^{(k)})$$

First consider solving the following system of nonlinear equations:

$$\begin{cases} f_1(x_1, x_2) = 0, \\ f_2(x_1, x_2) = 0. \end{cases}$$

Suppose $(x_1^{(k)}, x_2^{(k)})$ is an approximation to the solution of the system above, and we try to compute $h_1^{(k)}$ and $h_2^{(k)}$ such that $(x_1^{(k)} + h_1^{(k)}, x_2^{(k)} + h_2^{(k)})$ satisfies the system. By the Taylor's theorem for two variables,

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Put this in matrix form

$$\left[\begin{array}{cc} \frac{\partial f_1}{\partial x_1}(x_1^{(k)}, x_2^{(k)}) & \frac{\partial f_1}{\partial x_2}(x_1^{(k)}, x_2^{(k)}) \\ \frac{\partial f_2}{\partial x_1}(x_1^{(k)}, x_2^{(k)}) & \frac{\partial f_2}{\partial x_2}(x_1^{(k)}, x_2^{(k)}) \end{array} \right] \left[\begin{array}{c} h_1^{(k)} \\ h_2^{(k)} \end{array} \right] + \left[\begin{array}{c} f_1(x_1^{(k)}, x_2^{(k)}) \\ f_2(x_1^{(k)}, x_2^{(k)}) \end{array} \right] \approx \left[\begin{array}{c} 0 \\ 0 \end{array} \right].$$

$$J(x_1^{(k)}, x_2^{(k)}) \equiv \begin{bmatrix} \frac{\partial f_1}{\partial x_1} (x_1^{(k)}, x_2^{(k)}) & \frac{\partial f_1}{\partial x_2} (x_1^{(k)}, x_2^{(k)}) \\ \frac{\partial f_2}{\partial x_1} (x_1^{(k)}, x_2^{(k)}) & \frac{\partial f_2}{\partial x_2} (x_1^{(k)}, x_2^{(k)}) \end{bmatrix}$$

$$J(x_1^{(k)}, x_2^{(k)}) \begin{bmatrix} h_1^{(k)} \\ h_2^{(k)} \end{bmatrix} = - \begin{bmatrix} f_1(x_1^{(k)}, x_2^{(k)}) \\ f_2(x_1^{(k)}, x_2^{(k)}) \end{bmatrix},$$

$$\begin{bmatrix} x_1^{(k+1)} \\ x_1^{(k+1)} \\ x_2^{(k+1)} \end{bmatrix} = \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} + \begin{bmatrix} h_1^{(k)} \\ h_2^{(k)} \end{bmatrix}$$



Put this in matrix form

$$\left[\begin{array}{cc} \frac{\partial f_1}{\partial x_1}(x_1^{(k)}, x_2^{(k)}) & \frac{\partial f_1}{\partial x_2}(x_1^{(k)}, x_2^{(k)}) \\ \frac{\partial f_2}{\partial x_1}(x_1^{(k)}, x_2^{(k)}) & \frac{\partial f_2}{\partial x_2}(x_1^{(k)}, x_2^{(k)}) \end{array} \right] \left[\begin{array}{c} h_1^{(k)} \\ h_2^{(k)} \end{array} \right] + \left[\begin{array}{c} f_1(x_1^{(k)}, x_2^{(k)}) \\ f_2(x_1^{(k)}, x_2^{(k)}) \end{array} \right] \approx \left[\begin{array}{c} 0 \\ 0 \end{array} \right].$$

The matrix

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is called the Jacobian matrix. Set $h_1^{(k)}$ and $h_2^{(k)}$ be the solution of the linear system

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then

$$\begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ x_2^{(k)} \end{bmatrix} = \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} + \begin{bmatrix} h_1^{(k)} \\ h_2^{(k)} \end{bmatrix}$$



is expected to be a better approximation

Put this in matrix form

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1^{(k)}, x_2^{(k)}) & \frac{\partial f_1}{\partial x_2}(x_1^{(k)}, x_2^{(k)}) \\ \frac{\partial f_2}{\partial x_1}(x_1^{(k)}, x_2^{(k)}) & \frac{\partial f_2}{\partial x_2}(x_1^{(k)}, x_2^{(k)}) \end{bmatrix} \begin{bmatrix} h_1^{(k)} \\ h_2^{(k)} \end{bmatrix} + \begin{bmatrix} f_1(x_1^{(k)}, x_2^{(k)}) \\ f_2(x_1^{(k)}, x_2^{(k)}) \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

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is expected to be a better approximation.

$$x = \left[\begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array} \right]^T$$

and

$$F(x) = \begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \end{bmatrix}^T$$

The problem can be formulated as solving

$$F(x) = 0, \quad F: \mathbb{R}^n \to \mathbb{R}^n.$$

Let J(x), where the (i,j) entry is $\frac{\partial f_i}{\partial x_j}(x)$, be the $n \times n$ Jacobian matrix. Then the Newton's iteration is defined as

$$x^{(k+1)} = x^{(k)} + h^{(k)},$$

$$J(x^{(k)})h^{(k)} = -F(x^{(k)}).$$



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Algorithm (Newton's Method for Systems)

Given a function $F: \mathbb{R}^n \to \mathbb{R}^n$, an initial guess $x^{(0)}$ to the zero of F, and stop criteria M, δ , and ε , this algorithm performs the Newton's iteration to approximate one root of F.

Set
$$k=0$$
 and $h^{(-1)}=e_1$. While $(k < M)$ and $(\parallel h^{(k-1)} \parallel \ge \delta)$ and $(\parallel F(x^{(k)}) \parallel \ge \varepsilon)$ Calculate $J(x^{(k)}) = [\partial F_i(x^{(k)})/\partial x_j]$. Solve the $n \times n$ linear system $J(x^{(k)})h^{(k)} = -F(x^{(k)})$. Set $x^{(k+1)} = x^{(k)} + h^{(k)}$ and $k = k+1$.

End while

Output ("Convergent $x^{(k)}$ ") or ("Maximum number of iterations exceeded")



Let x^* be a solution of G(x) = x. Suppose $\exists \delta > 0$ with

- (i) $\partial g_i/\partial x_j$ is continuous on $N_\delta=\{x;\|x-x^*\|<\delta\}$ for all i and j.
- (ii) $\partial^2 g_i(x)/(\partial x_j \partial x_k)$ is continuous and

$$\left| \frac{\partial^2 g_i(x)}{\partial x_j \partial x_k} \right| \le M$$

for some M whenever $x \in N_{\delta}$ for each i, j and k.

(iii) $\partial g_i(x^*)/\partial x_k = 0$ for each i and k.

Then $\exists \ \hat{\delta} < \delta$ such that the sequence $\{x^{(k)}\}$ generated by

$$x^{(k)} = G(x^{(k-1)})$$

converges quadratically to x^* for any $x^{(0)}$ satisfying $\|x^{(0)}-x^*\|_{\infty}<\hat{\delta}$. Moreover,

$$||x^{(k)} - x^*||_{\infty} \le \frac{n^2 M}{2} ||x^{(k-1)} - x^*||_{\infty}^2, \forall k \ge 1.$$

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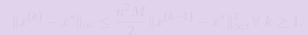
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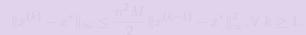
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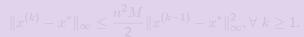
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Example

Consider the nonlinear system

$$3x_1 - \cos(x_2 x_3) - \frac{1}{2} = 0,$$

$$x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 = 0,$$

$$e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0.$$

Nonlinear functions: Let

$$F(x_1, x_2, x_3) = [f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3)]^T,$$

where

$$f_1(x_1, x_2, x_3) = 3x_1 - \cos(x_2 x_3) - \frac{1}{2},$$

 $f_2(x_1, x_2, x_3) = x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.6$
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• Nonlinear functions (cont.): The Jacobian matrix J(x) for this system is

$$J(x_1, x_2, x_3) = \begin{bmatrix} 3 & x_3 \sin x_2 x_3 & x_2 \sin x_2 x_3 \\ 2x_1 & -162(x_2 + 0.1) & \cos x_3 \\ -x_2 e^{-x_1 x_2} & -x_1 e^{-x_1 x_2} & 20 \end{bmatrix}.$$

• Newton's iteration with initial $x^{(0)} = [0.1, 0.1, -0.1]^T$:

$$\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{bmatrix} = \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{bmatrix} - \begin{bmatrix} h_1^{(k-1)} \\ h_2^{(k-1)} \\ h_3^{(k-1)} \end{bmatrix}.$$

where

$$\begin{bmatrix} h_1^{(k-1)} \\ h_2^{(k-1)} \\ h_3^{(k-1)} \end{bmatrix} = \left(J(x_1^{(k-1)}, x_2^{(k-1)}, x_3^{(k-1)}) \right)^{-1} F(x_1^{(k-1)}, x_2^{(k-1)}, x_3^{(k-1)})$$

4 D > 4 D > 4 D > 4 D > 90

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• Newton's iteration with initial $x^{(0)} = [0.1, 0.1, -0.1]^T$:

$$\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{bmatrix} = \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{bmatrix} - \begin{bmatrix} h_1^{(k-1)} \\ h_2^{(k-1)} \\ h_3^{(k-1)} \end{bmatrix},$$

where

$$\begin{bmatrix} h_1^{(k-1)} \\ h_2^{(k-1)} \\ h_3^{(k-1)} \end{bmatrix} = \left(J(x_1^{(k-1)}, x_2^{(k-1)}, x_3^{(k-1)}) \right)^{-1} F(x_1^{(k-1)}, x_2^{(k-1)}, x_3^{(k-1)}).$$

• Result:

\overline{k}	$x_1^{(k)}$	$x_{2}^{(k)}$	$x_{3}^{(k)}$	$ x^{(k)} - x^{(k-1)} _{\infty}$
0	0.10000000	0.10000000	-0.10000000	
1	0.50003702	0.01946686	-0.52152047	0.422
2	0.50004593	0.00158859	-0.52355711	1.79×10^{-2}
3	0.50000034	0.00001244	-0.52359845	1.58×10^{-3}
4	0.50000000	0.00000000	-0.52359877	1.24×10^{-5}
5	0.50000000	0.00000000	-0.52359877	0





Newton's Methods

- Advantage: quadratic convergence
- ▶ Disadvantage: For each iteration, it requires $O(n^3) + O(n^2) + O(n)$ arithmetic operations:
 - * n² partial derivatives for Jacobian matrix in most situations, the exact evaluation of the partial derivatives is inconvenient.
 - \star n scalar functional evaluations of F
 - ★ $O(n^3)$ arithmetic operations to solve linear system.
- quasi-Newton methods
 - Advantage: it requires only n scalar functional evaluations per iteration and $O(n^2)$ arithmetic operations
 - Disadvantage: superlinear convergence

Recall that in one dimensional case, one uses the linear model

$$\ell_k(x) = f(x_k) + a_k(x - x_k)$$

to approximate the function f(x) at x_k . That is, $\ell_k(x_k) = f(x_k)$ for $a_k \in \mathbb{R}$. If we further require that $\ell'(x_k) = f'(x_k)$, then $a_k \in \mathcal{F}(x_k)$.

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The zero of $\ell_k(x)$ is used to give a new approximate for the zero of f(x), that is,

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In multiple dimension, the analogue affine model becomes

$$M_k(x) = F(x_k) + A_k(x - x_k),$$

where $x, x_k \in \mathbb{R}^n$ and $A_k \in \mathbb{R}^{n \times n}$, and satisfies

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However, this secant equation can not uniquely determine A_k . One way of choosing A_k is to minimize M_k-M_{k-1} subject to the secant equation. Note

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For any $x \in \mathbb{R}^n$, we express

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for some $lpha \in \mathbb{R}$, $t_k \in \mathbb{R}^n$, and $h_k^T t_k = 0$. Then

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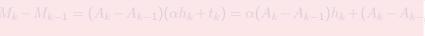
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both y_k and $A_{k-1}h_k$ are old values, we have no control over the first part $(A_k - A_{k-1})h_k$. In order to minimize $M_k(x) - M_{k-1}(x)$, we try to choose A_k so that

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Algorithm (Broyden's Method)

Given a n-variable nonlinear function $F: \mathbb{R}^n \to \mathbb{R}^n$, an initial iterate x_0 and initial Jacobian matrix $A_0 \in \mathbb{R}^{n \times n}$ (e.g., $A_0 = I$), this algorithm finds the solution for F(x) = 0.

Given x_0 , tolerance TOL, maximum number of iteration M.

Set
$$k = 1$$
.

While
$$k \leq M$$
 and $||x_k - x_{k-1}||_2 \geq TOL$

Solve
$$A_k h_{k+1} = -F(x_k)$$
 for h_{k+1}

Update
$$x_{k+1} = x_k + h_{k+1}$$

Compute
$$y_{k+1} = F(x_{k+1}) - F(x_k)$$

Update

$$A_{k+1} = A_k + \frac{(y_{k+1} - A_k h_{k+1}) h_{k+1}^T}{h_{k+1}^T h_{k+1}} = A_k + \frac{(y_{k+1} + F(x_k)) h_{k+1}^T}{h_{k+1}^T h_{k+1}}$$

$$k = k + 1$$

End While

- LU-factorization: cost $\frac{2}{3}n^3 + O(n^2)$ floating-point operations.
- Applying the Shermann-Morrison-Woodbury formula

$$(B + UV^{T})^{-1} = B^{-1} - B^{-1}U (I + V^{T}B^{-1}U)^{-1} V^{T}B^{-1}$$

$$A_{k}^{-1}$$

$$= \left[A_{k-1} + \frac{(y_{k} - A_{k-1}h_{k})h_{k}^{T}}{h_{k}^{T}h_{k}} \right]^{-1}$$

$$= A_{k-1}^{-1} - A_{k-1}^{-1} \frac{y_{k} - A_{k-1}h_{k}}{h_{k}^{T}h_{k}} \left(1 + h_{k}^{T}A_{k-1}^{-1} \frac{y_{k} - A_{k-1}h_{k}}{h_{k}^{T}h_{k}} \right)$$

$$= A_{k-1}^{-1} + \frac{(h_{k} - A_{k-1}^{-1}y_{k})h_{k}^{T}A_{k-1}^{-1}}{h_{k}^{T}A_{k-1}^{-1}y_{k}}.$$





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- LU-factorization: $\cos \frac{2}{3}n^3 + O(n^2)$ floating-point operations.
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Newton-based methods

- Advantage: high speed of convergence once a sufficiently accurate approximation
- Weakness: an accurate initial approximation to the solution is needed to ensure convergence.
- The Steepest Descent method converges only linearly to the solution but it will usually converge even for poor initial approximations.
- "Find sufficiently accurate starting approximate solution by using Steepest Descent method" + "Compute convergent solution by using Newton-based methods"
- The method of Steepest Descent determines a local minimum for a multivariable function of $g: \mathbb{R}^n \to \mathbb{R}$.
- A system of the form $f_i(x_1, ..., x_n) = 0$, i = 1, 2, ..., n has a solution at x iff the function g defined by

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If $g: \mathbb{R}^n \to \mathbb{R}$, the gradient, $\nabla g(x)$, at x is defined by

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Basic idea of steepest descent method:

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• Object: reduce g(x) to its minimal value zero. \Rightarrow for an initial approximation $x^{(0)}$, an appropriate choice for new vector $x^{(1)}$ is

$$x^{(1)} = x^{(0)} - \alpha \nabla g(x^{(0)}), \quad \text{ for some constant } \alpha > 0.$$

• Choose $\alpha > 0$ such that $g(x^{(1)}) < g(x^{(0)})$: define

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- lacktriangle Solve a root-finding problem $h'(lpha)=0 \; \Rightarrow \;$ Too costly, in general.
- ▶ Choose three number $\alpha_1 < \alpha_2 < \alpha_3$, construct quadratic polynomial P(x) that interpolates h at α_1, α_2 and α_3 , i.e.,

$$P(\alpha_1) = h(\alpha_1), \ P(\alpha_2) = h(\alpha_2), \ P(\alpha_3) = h(\alpha_3),$$

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Example

Use the Steepest Descent method with $x^{(0)} = (0,0,0)^T$ to find a reasonable starting approximation to the solution of the nonlinear system

$$f_1(x_1, x_2, x_3) = 3x_1 - \cos(x_2 x_3) - \frac{1}{2} = 0,$$

$$f_2(x_1, x_2, x_3) = x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 = 0,$$

$$f_3(x_1, x_2, x_3) = e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0.$$

Let
$$g(x1, x_2, x_3) = [f_1(x_1, x_2, x_3)]^2 + [f_2(x_1, x_2, x_3)]^2 + [f_3(x_1, x_2, x_3)]^2$$
.

Then
$$\nabla g(x_1, x_2, x_3) \equiv \nabla g(x)$$

$$= \left(2f_1(x)\frac{\partial f_1}{\partial x_1}(x) + 2f_2(x)\frac{\partial f_2}{\partial x_1}(x) + 2f_3(x)\frac{\partial f_3}{\partial x_1}(x), \right.$$

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$$g(x^{(0)}) = 111.975$$
 and $z_0 = \|\nabla g(x^{(0)})\|_2 = 419.554$.

$$z = \frac{1}{z_0} \nabla g(x^{(0)}) = [-0.0214514, -0.0193062, 0.999583]^T$$

$$g_1 = g(x^{(0)} - \alpha_1 z) = g(x^{(0)}) = 111.975.$$

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$$P(\alpha) = g_1 + h_1 \alpha + h_3 \alpha (\alpha - \alpha_2)$$

that interpolates $g(x^{(0)} - \alpha z)$ at $\alpha_1 = 0, \alpha_2 = 0.5$ and $\alpha_3 = 1$ as follows

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Thus

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so that

$$0 = P'(\alpha_0) = -419.346 + 801.872\alpha_0 \implies \alpha_0 = 0.522959$$

Since

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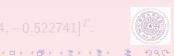
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