Final Exam

Jan 13, 2012, 13:10PM.

1. (12 pts $\times 4$) Consider solving the nonlinear scalar equation $\exp(x) - 0.99 = 0$.
   (a) Give a convergent fixed point iteration. Explain why it converges. Then start with $x_0 = 1$, report x_1 and x_2.
   (b) Give formula for the secant method. Then start with $x_0 = 2$, $x_1 = 1$, report x_2 and x_3.
   (c) Give formula for the Newton's method. Then start with $x_0 = 1$, report x_1 and x_2.
   (d) Give a cubically convergent iterative method. Then start with $x_0 = 1$, report x_1 and x_2.
2. (16 pts $\times 2$) Consider solving the nonlinear system of equations $\sin(\sin(x) + 2y) = 0.01$, $5x + \sin(6y) = 0.02$.
   (a) Give formula for the Newton's method. Then start with $(x_0, y_0) = (0, 0)$, report (x_1, y_1) and (x_2, y_2) and your final answer.
   (b) Give a convergent fixed point iteration. Explain why it converges. Then start with $(x_0, y_0) = (0, 0)$, report (x_1, y_1) and (x_2, y_2).
3. (12 pts $\times 3$) The following is a pseudo-code of Gauss-Siedel iteration for some linear system:

$$\begin{array}{l} u_0 = 0, \quad u_N = 0, \\ \ \text{do } k = 1, \cdots, \\ \quad \text{do } i = 1, \cdots, N-1, \\ \quad\ u_i = -0.5 * (u_{i-1} + u_{i+1}) + \sin(i\pi/N) \\ \quad \text{end } i \\ \text{end } k \end{array} \tag{1}$$

   (a) Write a pseudo-code for Jacobi and SOR methods for the same linear system.
   (b) Estimate the relative error of the numerical solution for $N = 1,000$. Explain (Need not implement).
   (c) Repeat (a) (only (a)) for the following system:

$$\begin{array}{l} u_{0,0} = u_{1,0} = \cdots = u_{N,0} = 0 \quad u_{0,N} = u_{1,N} = \cdots = u_{N,N} = 0 \\ u_{0,0} = u_{0,1} = \cdots = u_{0,N} = 0 \quad u_{N,0} = u_{N,1} = \cdots = u_{N,N} = 0 \\ \ \text{do } k = 1, \cdots, \\ \quad \text{do } j = 1, \cdots, N-1, \\ \quad \text{do } i = 1, \cdots, N-1, \\ \quad\ u_{i,j} = -0.25 * (u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}) + \sin(i\pi/N)\sin(j\pi/N) \\ \quad \text{end } i \\ \quad \text{end } j \\ \text{end } k \end{array} \tag{2}$$

4. (12 pts) Find the polynomial $p(x)$ that minimizes $\int_0^1 (e^x - p(x))^2 dx$ among all quadratic polynomials.
5. (12 pts) Derive Aitken's Δ^2 acceleration method. Then apply it to the sequence $a_n = (\frac{1}{2})^n$.