

Numerical Analysis I

Solutions of Equations in One Variable

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¹These slides are based on Prof. Tsung-Ming Huang(NTNU)'s original slides

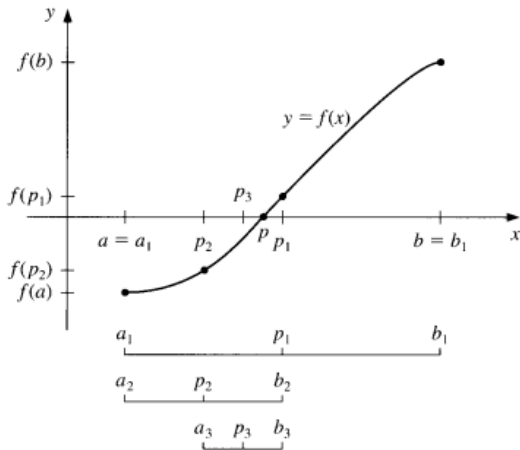
Outline



Bisection Method

Idea

If $f(x) \in C[a, b]$ and $f(a)f(b) < 0$, then $\exists c \in (a, b)$ such that $f(c) = 0$.



Bisection method algorithm

Given $f(x)$ defined on (a, b) , the maximal number of iterations M , and stop criteria δ and ε , this algorithm tries to locate one root of $f(x)$.

Compute $fa = f(a)$, $fb = f(b)$, and $e = b - a$

If $sign(fa) = sign(fb)$, **then** stop **End If**

For $k = 1, 2, \dots, M$

$e = e/2$, $c = (a + b)/2$, $fc = f(c)$

If $(|e| < \delta \text{ or } |fc| < \varepsilon)$, **then** stop **End If**

If $sign(fc) \neq sign(fa)$

$b = c$, $fb = fc$

Else

$a = c$, $fa = fc$

End If

End For



Let $\{c_n\}$ be the sequence of numbers produced. The algorithm should stop if one of the following conditions is satisfied.

- ❶ the iteration number $k > M$,
- ❷ $|c_k - c_{k-1}| < \delta$, or
- ❸ $|f(c_k)| < \varepsilon$.

Let $[a_0, b_0], [a_1, b_1], \dots$ denote the successive intervals produced by the bisection algorithm. Then

$$\begin{aligned} a &= a_0 \leq a_1 \leq a_2 \leq \dots \leq b_0 = b \\ \Rightarrow \quad &\{a_n\} \text{ and } \{b_n\} \text{ are bounded} \\ \Rightarrow \quad &\lim_{n \rightarrow \infty} a_n \text{ and } \lim_{n \rightarrow \infty} b_n \text{ exist} \end{aligned}$$



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Since

$$\begin{aligned}b_1 - a_1 &= \frac{1}{2}(b_0 - a_0) \\b_2 - a_2 &= \frac{1}{2}(b_1 - a_1) = \frac{1}{4}(b_0 - a_0) \\&\vdots \\b_n - a_n &= \frac{1}{2^n}(b_0 - a_0)\end{aligned}$$

hence

$$\lim_{n \rightarrow \infty} b_n - \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (b_n - a_n) = \lim_{n \rightarrow \infty} \frac{1}{2^n}(b_0 - a_0) = 0.$$

Therefore

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n \equiv z.$$

Since f is a continuous function, we have that

$$\lim_{n \rightarrow \infty} f(a_n) = f(\lim_{n \rightarrow \infty} a_n) = f(z) \quad \text{and} \quad \lim_{n \rightarrow \infty} f(b_n) = f(\lim_{n \rightarrow \infty} b_n) = f(z)$$



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On the other hand,

$$\begin{aligned} f(a_n)f(b_n) &< 0 \\ \Rightarrow \lim_{n \rightarrow \infty} f(a_n)f(b_n) &= f^2(z) \leq 0 \\ \Rightarrow f(z) &= 0 \end{aligned}$$

Therefore, the limit of the sequences $\{a_n\}$ and $\{b_n\}$ is a zero of f in $[a, b]$.
Let $c_n = \frac{1}{2}(a_n + b_n)$. Then

$$\begin{aligned} |z - c_n| &= \left| \lim_{n \rightarrow \infty} a_n - \frac{1}{2}(a_n + b_n) \right| \\ &= \left| \frac{1}{2} \left[\lim_{n \rightarrow \infty} a_n - b_n \right] + \frac{1}{2} \left[\lim_{n \rightarrow \infty} a_n - a_n \right] \right| \\ &\leq \max \left\{ \left| \lim_{n \rightarrow \infty} a_n - b_n \right|, \left| \lim_{n \rightarrow \infty} a_n - a_n \right| \right\} \\ &\leq |b_n - a_n| = \frac{1}{2^n} |b_0 - a_0|. \end{aligned}$$

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Example

How many steps should be taken to compute a root of $f(x) = x^3 + 4x^2 - 10 = 0$ on $[1, 2]$ with relative error 10^{-3} ?

solution: Seek an n such that

$$\frac{|z - c_n|}{|z|} \leq 10^{-3} \Rightarrow |z - c_n| \leq |z| \times 10^{-3}.$$

Since $z \in [1, 2]$, it is sufficient to show

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Fixed-Point Iteration

Definition

x is called a **fixed point** of a given function f if $f(x) = x$.

Root-finding problems and fixed-point problems

- Find x^* such that $f(x^*) = 0$.

Let $g(x) = x - f(x)$. Then $g(x^*) = x^* - f(x^*) = x^*$.

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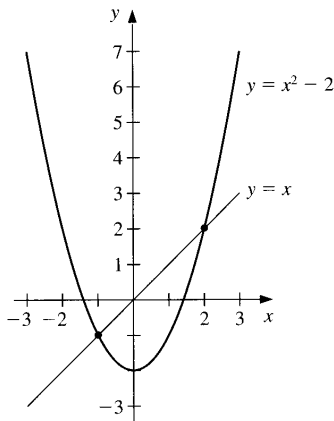
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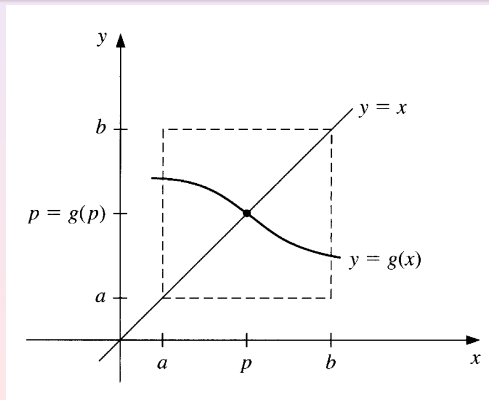
The function $g(x) = x^2 - 2$, for $-2 \leq x \leq 3$, has fixed points at $x = -1$ and $x = 2$ since

$$g(-1) = (-1)^2 - 2 = -1 \quad \text{and} \quad g(2) = 2^2 - 2 = 2.$$



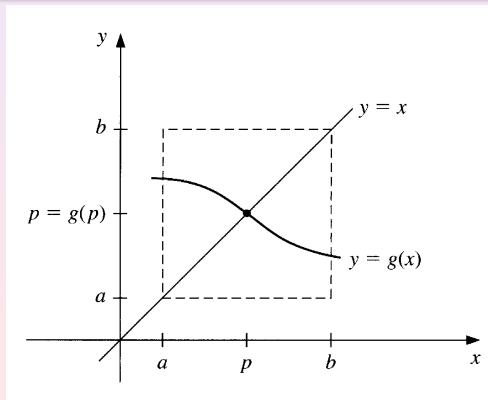
Theorem (Existence and uniqueness)

- 1 If $g \in C[a, b]$ such that $a \leq g(x) \leq b$ for all $x \in [a, b]$, then g has a fixed point in $[a, b]$.
- 2 If, in addition, $g'(x)$ exists in (a, b) and there exists a positive constant $M < 1$ such that $|g'(x)| \leq M < 1$ for all $x \in (a, b)$. Then the fixed point is unique.



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Proof

Existence:

- If $g(a) = a$ or $g(b) = b$, then a or b is a fixed point of g and we are done.
- Otherwise, it must be $g(a) > a$ and $g(b) < b$. The function $h(x) = g(x) - x$ is continuous on $[a, b]$, with

$$h(a) = g(a) - a > 0 \quad \text{and} \quad h(b) = g(b) - b < 0.$$

By the Intermediate Value Theorem, $\exists x^* \in [a, b]$ such that $h(x^*) = 0$. That is

$$g(x^*) - x^* = 0 \Rightarrow g(x^*) = x^*.$$

Hence g has a fixed point x^* in $[a, b]$.



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Uniqueness:

Suppose that $p \neq q$ are both fixed points of g in $[a, b]$. By the Mean-Value theorem, there exists ξ between p and q such that

$$g'(\xi) = \frac{g(p) - g(q)}{p - q} = \frac{p - q}{p - q} = 1.$$

However, this contradicts to the assumption that $|g'(x)| \leq M < 1$ for all x in $[a, b]$. Therefore the fixed point of g is unique. \square



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Example

Show that the following function has a unique fixed point.

$$g(x) = (x^2 - 1)/3, \quad x \in [-1, 1].$$

Solution: The Extreme Value Theorem implies that

$$\begin{aligned} \min_{x \in [-1, 1]} g(x) &= g(0) = -\frac{1}{3}, \\ \max_{x \in [-1, 1]} g(x) &= g(\pm 1) = 0. \end{aligned}$$

That is $g(x) \in [-1, 1]$, $\forall x \in [-1, 1]$.

Moreover, g is continuous and

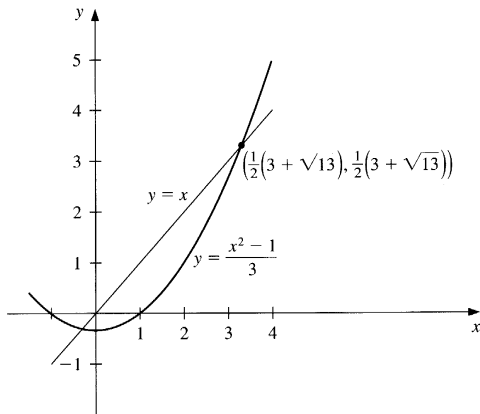
$$|g'(x)| = \left| \frac{2x}{3} \right| \leq \frac{2}{3}, \quad \forall x \in (-1, 1).$$

By above theorem, g has a unique fixed point in $[-1, 1]$.



Let p be such unique fixed point of g . Then

$$\begin{aligned} p = g(p) &= \frac{p^2 - 1}{3} \Rightarrow p^2 - 3p - 1 = 0 \\ &\Rightarrow p = \frac{1}{2}(3 + \sqrt{13}). \end{aligned}$$



Fixed-point iteration or functional iteration

Given a continuous function g , choose an initial point x_0 and generate $\{x_k\}_{k=0}^{\infty}$ by

$$x_{k+1} = g(x_k), \quad k \geq 0.$$

$\{x_k\}$ may not converge, e.g., $g(x) = 3x$. However, when the sequence converges, say,

$$\lim_{k \rightarrow \infty} x_k = x^*,$$

then, since g is continuous,

$$g(x^*) = g\left(\lim_{k \rightarrow \infty} x_k\right) = \lim_{k \rightarrow \infty} g(x_k) = \lim_{k \rightarrow \infty} x_{k+1} = x^*.$$

That is, x^* is a fixed point of g .



Fixed-point iteration

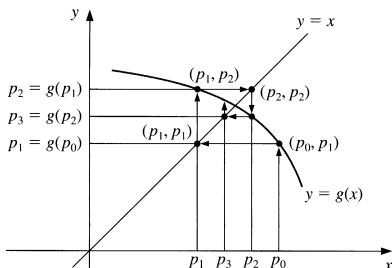
Given x_0 , tolerance TOL , maximum number of iteration M .

Set $i = 1$ and $x = g(x_0)$.

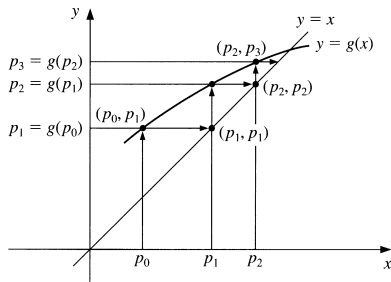
While $i \leq M$ and $|x - x_0| \geq TOL$

Set $i = i + 1$, $x_0 = x$ and $x = g(x_0)$.

End While



(a)



(b)

Example

The equation

$$x^3 + 4x^2 - 10 = 0$$

has a unique root in $[1, 2]$. Change the equation to the fixed-point form $x = g(x)$.

$$(a) \ x = g_1(x) \equiv x - f(x) = x - x^3 - 4x^2 + 10$$

$$(b) \ x = g_2(x) = \left(\frac{10}{x} - 4x\right)^{1/2}$$

$$x^3 = 10 - 4x^2 \Rightarrow x^2 = \frac{10}{x} - 4x \Rightarrow x = \pm \left(\frac{10}{x} - 4x\right)^{1/2}$$



$$(c) \ x = g_3(x) = \frac{1}{2} (10 - x^3)^{1/2}$$

$$4x^2 = 10 - x^3 \Rightarrow x = \pm \frac{1}{2} (10 - x^3)^{1/2}$$

$$(d) \ x = g_4(x) = \left(\frac{10}{4+x} \right)^{1/2}$$

$$x^2(x+4) = 10 \Rightarrow x = \pm \left(\frac{10}{4+x} \right)^{1/2}$$

$$(e) \ x = g_5(x) = x - \frac{x^3+4x^2-10}{3x^2+8x}$$

$$x = g_5(x) \equiv x - \frac{f(x)}{f'(x)}$$



Results of the fixed-point iteration with initial point $x_0 = 1.5$

n	(a)	(b)	(c)	(d)	(e)
0	1.5	1.5	1.5	1.5	1.5
1	-0.875	0.8165	1.286953768	1.348399725	1.373333333
2	6.732	2.9969	1.402540804	1.367376372	1.365262015
3	-469.7	$(-8.65)^{1/2}$	1.345458374	1.364957015	1.365230014
4	1.03×10^8		1.375170253	1.365264748	1.365230013
5			1.360094193	1.365225594	
6			1.367846968	1.365230576	
7			1.363887004	1.365229942	
8			1.365916734	1.365230022	
9			1.364878217	1.365230012	
10			1.365410062	1.365230014	
15			1.365223680	1.365230013	
20			1.365230236		
25			1.365230006		
30			1.365230013		



Theorem (Fixed-point Theorem)

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$ for all $x \in [a, b]$. Suppose that g' exists on (a, b) and that $\exists k$ with $0 < k < 1$ such that

$$|g'(x)| \leq k, \quad \forall x \in (a, b).$$

Then, for any number x_0 in $[a, b]$,

$$x_n = g(x_{n-1}), \quad n \geq 1,$$

converges to the unique fixed point x in $[a, b]$.



Proof: By the assumptions, a unique fixed point exists in $[a, b]$. Since $g([a, b]) \subseteq [a, b]$, $\{x_n\}_{n=0}^{\infty}$ is defined and $x_n \in [a, b]$ for all $n \geq 0$. Using the Mean Values Theorem and the fact that $|g'(x)| \leq k$, we have

$$|x - x_n| = |g(x_{n-1}) - g(x)| = |g'(\xi_n)| |x - x_{n-1}| \leq k |x - x_{n-1}|,$$

where $\xi_n \in (a, b)$. It follows that

$$|x_n - x| \leq k |x_{n-1} - x| \leq k^2 |x_{n-2} - x| \leq \cdots \leq k^n |x_0 - x|. \quad (1)$$

Since $0 < k < 1$, we have

$$\lim_{n \rightarrow \infty} k^n = 0$$

and

$$\lim_{n \rightarrow \infty} |x_n - x| \leq \lim_{n \rightarrow \infty} k^n |x_0 - x| = 0.$$

Hence, $\{x_n\}_{n=0}^{\infty}$ converges to x .



Corollary

If g satisfies the hypotheses of above theorem, then

$$|x - x_n| \leq k^n \max\{x_0 - a, b - x_0\}$$

and

$$|x_n - x| \leq \frac{k^n}{1 - k} |x_1 - x_0|, \quad \forall n \geq 1.$$

Proof: From (1),

$$|x_n - x| \leq k^n |x_0 - x| \leq k^n \max\{x_0 - a, b - x_0\}.$$

For $n \geq 1$, using the Mean Values Theorem,

$$|x_{n+1} - x_n| = |g(x_n) - g(x_{n-1})| \leq k |x_n - x_{n-1}| \leq \cdots \leq k^n |x_1 - x_0|.$$



Thus, for $m > n \geq 1$,

$$\begin{aligned} |x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - \cdots + x_{n+1} - x_n| \\ &\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\ &\leq k^{m-1}|x_1 - x_0| + k^{m-2}|x_1 - x_0| + \cdots + k^n|x_1 - x_0| \\ &= k^n|x_1 - x_0| (1 + k + k^2 + \cdots + k^{m-n-1}). \end{aligned}$$

It implies that

$$\begin{aligned} |x - x_n| &= \lim_{m \rightarrow \infty} |x_m - x_n| \leq \lim_{m \rightarrow \infty} k^n |x_1 - x_0| \sum_{j=0}^{m-n-1} k^j \\ &\leq k^n |x_1 - x_0| \sum_{j=0}^{\infty} k^j = \frac{k^n}{1 - k} |x_1 - x_0|. \end{aligned}$$



Example

For previous example, $f(x) = x^3 + 4x^2 - 10 = 0$.

Let $g_1(x) = x - x^3 - 4x^2 + 10$, we have

$$g_1(1) = 6 \quad \text{and} \quad g_1(2) = -12,$$

so $g_1([1, 2]) \not\subseteq [1, 2]$. Moreover,

$$g_1'(x) = 1 - 3x^2 - 8x \Rightarrow |g_1'(x)| \geq 1 \quad \forall x \in [1, 2]$$

Convergence is NOT guaranteed. In fact, it almost for sure will not converge since when x_n is close to the solution x^* ,

$$|x_n - x^*| = |g_1(x_{n-1}) - g_1(x^*)| = |g_1'(c)(x_{n-1} - x^*)| > |x_{n-1} - x^*|.$$

The error is amplified whenever x_n is close to convergence. The only possibility for convergence is when x_n is far from x^* and (by chance, and very unlikely) that $g_1(x_n) = x^*$.

For $g_3(x) = \frac{1}{2}(10 - x^3)^{1/2}$, $\forall x \in [1, 1.5]$,

$$g_3'(x) = -\frac{3}{4}x^2(10 - x^3)^{-1/2} < 0, \forall x \in [1, 1.5],$$

so g_3 is strictly decreasing on $[1, 1.5]$ and

$$1 < 1.28 \approx g_3(1.5) \leq g_3(x) \leq g_3(1) = 1.5, \forall x \in [1, 1.5].$$

On the other hand,

$$|g_3'(x)| \leq |g_3'(1.5)| \approx 0.66, \forall x \in [1, 1.5]$$

Hence, the sequence is convergent to the fixed point.



For $g_4(x) = \sqrt{10/(4+x)}$, we have

$$\sqrt{\frac{10}{6}} \leq g_4(x) \leq \sqrt{\frac{10}{5}}, \quad \forall x \in [1, 2] \quad \Rightarrow \quad g_4([1, 2]) \subseteq [1, 2]$$

Moreover,

$$|g'_4(x)| = \left| \frac{-5}{\sqrt{10}(4+x)^{3/2}} \right| \leq \frac{5}{\sqrt{10}(5)^{3/2}} < 0.15, \quad \forall x \in [1, 2].$$

The bound of $|g'_4(x)|$ is much smaller than the bound of $|g'_3(x)|$, which explains the more rapid convergence using g_4 .



Newton's method

Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ and $f \in C^2[a, b]$, i.e., f'' exists and is continuous. If $f(x^*) = 0$ and $x^* = x + h$ where h is small, then by Taylor's theorem

$$\begin{aligned} 0 = f(x^*) &= f(x + h) \\ &= f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{3!}f'''(x)h^3 + \cdots \\ &= f(x) + f'(x)h + O(h^2). \end{aligned}$$

Since h is small, $O(h^2)$ is negligible. It is reasonable to drop $O(h^2)$ terms. This implies

$$f(x) + f'(x)h \approx 0 \quad \text{and} \quad h \approx -\frac{f(x)}{f'(x)}, \quad \text{if } f'(x) \neq 0.$$

Hence

$$x + h = x - \frac{f(x)}{f'(x)}$$

is a better approximation to x^* .



This sets the stage for the **Newton-Rapbson's** method, which starts with an initial approximation x_0 and generates the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Since the Taylor's expansion of $f(x)$ at x_k is given by

$$f(x) = f(x_k) + f'(x_k)(x - x_k) + \frac{1}{2}f''(x_k)(x - x_k)^2 + \cdots.$$

At x_k , one uses the **tangent line**

$$y = \ell(x) = f(x_k) + f'(x_k)(x - x_k)$$

to **approximate the curve** of $f(x)$ and uses the zero of the tangent line to approximate the zero of $f(x)$.



Newton's Method

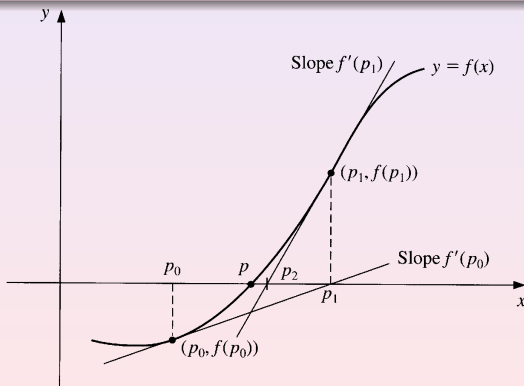
Given x_0 , tolerance TOL , maximum number of iteration M .

Set $i = 1$ and $x = x_0 - f(x_0)/f'(x_0)$.

While $i \leq M$ and $|x - x_0| \geq TOL$

Set $i = i + 1$, $x_0 = x$ and $x = x_0 - f(x_0)/f'(x_0)$.

End While



Three stopping-technique inequalities

$$(a). \quad |x_n - x_{n-1}| < \varepsilon,$$

$$(b). \quad \frac{|x_n - x_{n-1}|}{|x_n|} < \varepsilon, \quad x_n \neq 0,$$

$$(c). \quad |f(x_n)| < \varepsilon.$$

Note that Newton's method for solving $f(x) = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{for } n \geq 1$$

is just a special case of functional iteration in which

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Example

The following table shows the convergence behavior of Newton's method applied to solving $f(x) = x^2 - 1 = 0$. Observe the quadratic convergence rate.

n	x_n	$ e_n \equiv 1 - x_n $
0	2.0	1
1	1.25	0.25
2	1.025	2.5e-2
3	1.0003048780488	3.048780488e-4
4	1.0000000464611	4.64611e-8
5	1.0	0



Theorem

Assume $f(x^*) = 0$, $f'(x^*) \neq 0$ and $f(x)$, $f'(x)$ and $f''(x)$ are continuous on $N_\varepsilon(x^*)$. Then if x_0 is chosen sufficiently close to x^* , then

$$\left\{ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \right\} \rightarrow x^*.$$

Proof: Define

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Find an interval $[x^* - \delta, x^* + \delta]$ such that

$$g([x^* - \delta, x^* + \delta]) \subseteq [x^* - \delta, x^* + \delta]$$

and

$$|g'(x)| \leq k < 1, \quad \forall x \in (x^* - \delta, x^* + \delta).$$



Since f' is continuous and $f'(x^*) \neq 0$, it implies that $\exists \delta_1 > 0$ such that $f'(x) \neq 0 \forall x \in [x^* - \delta_1, x^* + \delta_1] \subseteq [a, b]$. Thus, g is defined and continuous on $[x^* - \delta_1, x^* + \delta_1]$. Also

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2},$$

for $x \in [x^* - \delta_1, x^* + \delta_1]$. Since f'' is continuous on $[a, b]$, we have g' is continuous on $[x^* - \delta_1, x^* + \delta_1]$.

By assumption $f(x^*) = 0$, so

$$g'(x^*) = \frac{f(x^*)f''(x^*)}{|f'(x^*)|^2} = 0.$$

Since g' is continuous on $[x^* - \delta_1, x^* + \delta_1]$ and $g'(x^*) = 0$, $\exists \delta$ with $0 < \delta < \delta_1$ and $k \in (0, 1)$ such that

$$|g'(x)| \leq k, \forall x \in [x^* - \delta, x^* + \delta].$$



Claim: $g([x^* - \delta, x^* + \delta]) \subseteq [x^* - \delta, x^* + \delta]$.

If $x \in [x^* - \delta, x^* + \delta]$, then, by the Mean Value Theorem, $\exists \xi$ between x and x^* such that

$$|g(x) - g(x^*)| = |g'(\xi)||x - x^*|.$$

It implies that

$$\begin{aligned} |g(x) - x^*| &= |g(x) - g(x^*)| = |g'(\xi)||x - x^*| \\ &\leq k|x - x^*| < |x - x^*| < \delta. \end{aligned}$$

Hence, $g([x^* - \delta, x^* + \delta]) \subseteq [x^* - \delta, x^* + \delta]$.

By the Fixed-Point Theorem, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

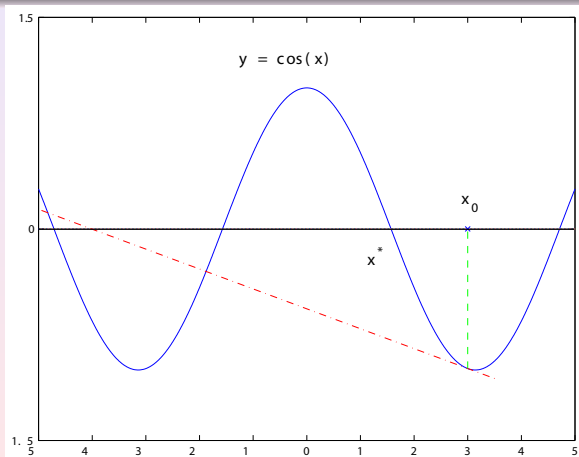
$$x_n = g(x_{n-1}) = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}, \quad \text{for } n \geq 1,$$

converges to x^* for any $x_0 \in [x^* - \delta, x^* + \delta]$.



Example

When Newton's method applied to $f(x) = \cos x$ with starting point $x_0 = 3$, which is close to the root $\frac{\pi}{2}$ of f , it produces $x_1 = -4.01525$, $x_2 = -4.8526, \dots$, which converges to another root $-\frac{3\pi}{2}$.



Secant method

Disadvantage of Newton's method

In many applications, the derivative $f'(x)$ is very expensive to compute, or the function $f(x)$ is not given in an algebraic formula so that $f'(x)$ is not available.

By definition,

$$f'(x_{n-1}) = \lim_{x \rightarrow x_{n-1}} \frac{f(x) - f(x_{n-1})}{x - x_{n-1}}.$$

Letting $x = x_{n-2}$, we have

$$f'(x_{n-1}) \approx \frac{f(x_{n-2}) - f(x_{n-1})}{x_{n-2} - x_{n-1}} = \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}}.$$

Using this approximation for $f'(x_{n-1})$ in Newton's formula gives

$$x_n = x_{n-1} - \frac{f(x_{n-1})(x_{n-1} - x_{n-2})}{f(x_{n-1}) - f(x_{n-2})},$$

which is called the **Secant method**.



From geometric point of view, we use a **secant line** through x_{n-1} and x_{n-2} instead of the tangent line to approximate the function at the point x_{n-1} . The slope of the secant line is

$$s_{n-1} = \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}}$$

and the equation is

$$M(x) = f(x_{n-1}) + s_{n-1}(x - x_{n-1}).$$

The zero of the secant line

$$x = x_{n-1} - \frac{f(x_{n-1})}{s_{n-1}} = x_{n-1} - f(x_{n-1}) \frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}$$

is then used as a new approximate x_n .



Secant Method

Given x_0, x_1 , tolerance TOL , maximum number of iteration M .

Set $i = 2$; $y_0 = f(x_0)$; $y_1 = f(x_1)$;

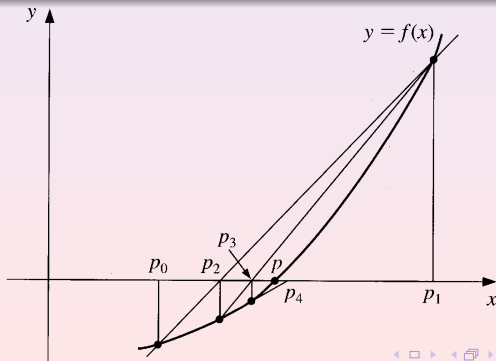
$$x = x_1 - y_1(x_1 - x_0)/(y_1 - y_0).$$

While $i \leq M$ and $|x - x_1| \geq TOL$

Set $i = i + 1$; $x_0 = x_1$; $y_0 = y_1$; $x_1 = x$; $y_1 = f(x)$;

$$x = x_1 - y_1(x_1 - x_0)/(y_1 - y_0).$$

End While



Method of False Position

- 1 Choose initial approximations x_0 and x_1 with $f(x_0)f(x_1) < 0$.
- 2 $x_2 = x_1 - f(x_1)(x_1 - x_0)/(f(x_1) - f(x_0))$
- 3 Decide which secant line to use to compute x_3 :

If $f(x_2)f(x_1) < 0$, then x_1 and x_2 bracket a root, i.e.,

$$x_3 = x_2 - f(x_2)(x_2 - x_1)/(f(x_2) - f(x_1))$$

Else, x_0 and x_2 bracket a root, i.e.,

$$x_3 = x_2 - f(x_2)(x_2 - x_0)/(f(x_2) - f(x_0))$$

End if



Method of False Position

Given x_0, x_1 , tolerance TOL , maximum number of iteration M .

Set $i = 2$; $y_0 = f(x_0)$; $y_1 = f(x_1)$; $x = x_1 - y_1(x_1 - x_0)/(y_1 - y_0)$.

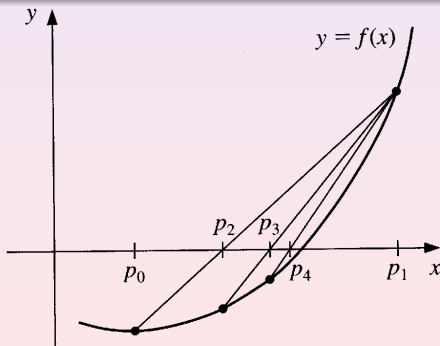
While $i \leq M$ and $|x - x_1| \geq TOL$

Set $i = i + 1$; $y = f(x)$.

If $y \cdot y_1 < 0$, then set $x_0 = x$; $y_0 = y_1$.

Set $x_1 = x$; $y_1 = y$; $x = x_1 - y_1(x_1 - x_0)/(y_1 - y_0)$.

End While



Error analysis for iterative methods

Definition

Let $\{x_n\} \rightarrow x^*$. If there are positive constants c and α such that

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^\alpha} = c,$$

then we say the **rate of convergence** is of **order** α .

We say that the rate of convergence is

① **linear** if $\alpha = 1$ and $0 < c < 1$.

② **superlinear** if

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|} = 0;$$

③ **quadratic** if $\alpha = 2$.

Suppose that $\{x_n\}_{n=0}^{\infty}$ and $\{\tilde{x}_n\}_{n=0}^{\infty}$ are linearly and quadratically convergent to x^* , respectively, with the same constant $c = 0.5$. For simplicity, suppose that

$$\frac{|x_{n+1} - x^*|}{|x_n - x^*|} \approx c \quad \text{and} \quad \frac{|\tilde{x}_{n+1} - x^*|}{|\tilde{x}_n - x^*|^2} \approx c.$$

These imply that

$$|x_n - x^*| \approx c|x_{n-1} - x^*| \approx c^2|x_{n-2} - x^*| \approx \cdots \approx c^n|x_0 - x^*|,$$

and

$$\begin{aligned} |\tilde{x}_n - x^*| &\approx c|\tilde{x}_{n-1} - x^*|^2 \approx c \left[c|\tilde{x}_{n-2} - x^*|^2 \right]^2 = c^3|\tilde{x}_{n-2} - x^*|^4 \\ &\approx c^3 \left[c|\tilde{x}_{n-3} - x^*|^2 \right]^4 = c^7|\tilde{x}_{n-3} - x^*|^8 \\ &\approx \cdots \approx c^{2^n-1}|\tilde{x}_0 - x^*|^{2^n}. \end{aligned}$$



Remark

Quadratically convergent sequences generally converge much more quickly than those that converge only linearly.

Theorem

Let $g \in C[a, b]$ with $g([a, b]) \subseteq [a, b]$. Suppose that g' is continuous on (a, b) and $\exists k \in (0, 1)$ such that

$$|g'(x)| \leq k, \quad \forall x \in (a, b).$$

If $g'(x^*) \neq 0$, then for any $x_0 \in [a, b]$, the sequence

$$x_n = g(x_{n-1}), \quad \text{for } n \geq 1$$

converges only linearly to the unique fixed point x^* in $[a, b]$.



Proof:

- By the Fixed-Point Theorem, the sequence $\{x_n\}_{n=0}^{\infty}$ converges to x^* .
- Since g' exists on (a, b) , by the Mean Value Theorem, $\exists \xi_n$ between x_n and x^* such that

$$x_{n+1} - x^* = g(x_n) - g(x^*) = g'(\xi_n)(x_n - x^*).$$

- $\because \{x_n\}_{n=0}^{\infty} \rightarrow x^* \Rightarrow \{\xi_n\}_{n=0}^{\infty} \rightarrow x^*$
- Since g' is continuous on (a, b) , we have

$$\lim_{n \rightarrow \infty} g'(\xi_n) = g'(x^*).$$

- Thus,

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|} = \lim_{n \rightarrow \infty} |g'(\xi_n)| = |g'(x^*)|.$$

Hence, if $g'(x^*) \neq 0$, fixed-point iteration exhibits linear convergence.



Theorem

Let x^* be a fixed point of g and I be an open interval with $x^* \in I$. Suppose that $g'(x^*) = 0$ and g'' is continuous with

$$|g''(x)| < M, \quad \forall x \in I.$$

Then $\exists \delta > 0$ such that

$$\{x_n = g(x_{n-1})\}_{n=1}^{\infty} \rightarrow x^* \quad \text{for } x_0 \in [x^* - \delta, x^* + \delta]$$

at least quadratically. Moreover,

$$|x_{n+1} - x^*| < \frac{M}{2} |x_n - x^*|^2, \quad \text{for sufficiently large } n.$$



Proof:

- Since $g'(x^*) = 0$ and g' is continuous on I , $\exists \delta$ such that $[x^* - \delta, x^* + \delta] \subset I$ and

$$|g'(x)| \leq k < 1, \quad \forall x \in [x^* - \delta, x^* + \delta].$$

- In the proof of the convergence for Newton's method, we have

$$\{x_n\}_{n=0}^{\infty} \subset [x^* - \delta, x^* + \delta].$$

- Consider the Taylor expansion of $g(x_n)$ at x^*

$$\begin{aligned} x_{n+1} = g(x_n) &= g(x^*) + g'(x^*)(x_n - x^*) + \frac{g''(\xi)}{2}(x_n - x^*)^2 \\ &= x^* + \frac{g''(\xi)}{2}(x_n - x^*)^2, \end{aligned}$$

where ξ lies between x_n and x^* .



- Since

$$|g'(x)| \leq k < 1, \forall x \in [x^* - \delta, x^* + \delta]$$

and

$$g([x^* - \delta, x^* + \delta]) \subseteq [x^* - \delta, x^* + \delta],$$

it follows that $\{x_n\}_{n=0}^{\infty}$ converges to x^* .

- But ξ_n is between x_n and x^* for each n , so $\{\xi_n\}_{n=0}^{\infty}$ also converges to x^* and

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^2} = \frac{|g''(x^*)|}{2} < \frac{M}{2}.$$

- It implies that $\{x_n\}_{n=0}^{\infty}$ is quadratically convergent to x^* if $g''(x^*) \neq 0$ and

$$|x_{n+1} - x^*| < \frac{M}{2} |x_n - x^*|^2, \text{ for sufficiently large } n. \quad \square$$



For Newton's method,

$$g(x) = x - \frac{f(x)}{f'(x)} \Rightarrow g'(x) = 1 - \frac{f'(x)}{f'(x)} + \frac{f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2}$$

It follows that $g'(x^*) = 0$. Hence Newton's method is locally quadratically convergent.



Error Analysis of Secant Method

Reference: D. Kincaid and W. Cheney, "Numerical analysis"

Let x^* denote the exact solution of $f(x) = 0$, $e_k = x_k - x^*$ be the errors at the k -th step. Then

$$\begin{aligned}e_{k+1} &= x_{k+1} - x^* \\&= x_k - f(x_k) \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} - x^* \\&= \frac{1}{f(x_k) - f(x_{k-1})} [(x_{k-1} - x^*)f(x_k) - (x_k - x^*)f(x_{k-1})] \\&= \frac{1}{f(x_k) - f(x_{k-1})} (e_{k-1}f(x_k) - e_k f(x_{k-1})) \\&= e_k e_{k-1} \left(\frac{\frac{1}{e_k} f(x_k) - \frac{1}{e_{k-1}} f(x_{k-1})}{x_k - x_{k-1}} \cdot \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \right)\end{aligned}$$



To estimate the numerator $\frac{\frac{1}{e_k}f(x_k) - \frac{1}{e_{k-1}}f(x_{k-1})}{x_k - x_{k-1}}$, we apply the Taylor's theorem

$$f(x_k) = f(x^* + e_k) = f(x^*) + f'(x^*)e_k + \frac{1}{2}f''(x^*)e_k^2 + O(e_k^3),$$

to get

$$\frac{1}{e_k}f(x_k) = f'(x^*) + \frac{1}{2}f''(x^*)e_k + O(e_k^2).$$

Similarly,

$$\frac{1}{e_{k-1}}f(x_{k-1}) = f'(x^*) + \frac{1}{2}f''(x^*)e_{k-1} + O(e_{k-1}^2).$$

Hence

$$\frac{1}{e_k}f(x_k) - \frac{1}{e_{k-1}}f(x_{k-1}) \approx \frac{1}{2}(e_k - e_{k-1})f''(x^*).$$

Since $x_k - x_{k-1} = e_k - e_{k-1}$ and

$$\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \rightarrow \frac{1}{f'(x^*)},$$



we have

$$\begin{aligned} e_{k+1} &\approx e_k e_{k-1} \left(\frac{\frac{1}{2}(e_k - e_{k-1})f''(x^*)}{e_k - e_{k-1}} \cdot \frac{1}{f'(x^*)} \right) = \frac{1}{2} \frac{f''(x^*)}{f'(x^*)} e_k e_{k-1} \\ &\equiv C e_k e_{k-1} \end{aligned} \quad (2)$$

To estimate the convergence rate, we assume

$$|e_{k+1}| \approx \eta |e_k|^\alpha,$$

where $\eta > 0$ and $\alpha > 0$ are constants, i.e.,

$$\frac{|e_{k+1}|}{\eta |e_k|^\alpha} \rightarrow 1 \quad \text{as } k \rightarrow \infty.$$

Then $|e_k| \approx \eta |e_{k-1}|^\alpha$ which implies $|e_{k-1}| \approx \eta^{-1/\alpha} |e_k|^{1/\alpha}$. Hence (2) gives

$$\eta |e_k|^\alpha \approx C |e_k| \eta^{-1/\alpha} |e_k|^{1/\alpha} \implies C^{-1} \eta^{1+\frac{1}{\alpha}} \approx |e_k|^{1-\alpha+\frac{1}{\alpha}}.$$

Since $|e_k| \rightarrow 0$ as $k \rightarrow \infty$, and $C^{-1} \eta^{1+\frac{1}{\alpha}}$ is a nonzero constant,

$$1 - \alpha + \frac{1}{\alpha} = 0 \implies \alpha = \frac{1 + \sqrt{5}}{2} \approx 1.62.$$



This result implies that $C^{-1}\eta^{1+\frac{1}{\alpha}} \rightarrow 1$ and

$$\eta \rightarrow C^{\frac{\alpha}{1+\alpha}} = \left(\frac{f''(x^*)}{2f'(x^*)} \right)^{0.62}.$$

In summary, we have shown that

$$|e_{k+1}| = \eta |e_k|^\alpha, \quad \alpha \approx 1.62,$$

that is, the rate of convergence is superlinear.

Rate of convergence:

- secant method: superlinear
- Newton's method: quadratic
- bisection method: linear



Each iteration of method requires

- secant method: one function evaluation
- Newton's method: two function evaluation, namely, $f(x_k)$ and $f'(x_k)$.
 \Rightarrow two steps of secant method are comparable to one step of Newton's method. Thus

$$|e_{k+2}| \approx \eta |e_{k+1}|^\alpha \approx \eta^{1+\alpha} |e_k|^{\frac{3+\sqrt{5}}{2}} \approx \eta^{1+\alpha} |e_k|^{2.62}.$$

\Rightarrow secant method is more efficient than Newton's method.

Remark

Two steps of secant method would require a little more work than one step of Newton's method.



Accelerating convergence

Aitken's Δ^2 method

- Accelerate the convergence of a sequence that is **linearly convergent**.
- Suppose $\{y_n\}_{n=0}^{\infty}$ is a linearly convergent sequence with limit y . Construct a sequence $\{\hat{y}_n\}_{n=0}^{\infty}$ that converges more rapidly to y than $\{y_n\}_{n=0}^{\infty}$.

For n sufficiently large,

$$\frac{y_{n+1} - y}{y_n - y} \approx \frac{y_{n+2} - y}{y_{n+1} - y}.$$

Then

$$(y_{n+1} - y)^2 \approx (y_{n+2} - y)(y_n - y),$$

so

$$y_{n+1}^2 - 2y_{n+1}y + y^2 \approx y_{n+2}y_n - (y_{n+2} + y_n)y + y^2$$



and

$$(y_{n+2} + y_n - 2y_{n+1})y \approx y_{n+2}y_n - y_{n+1}^2.$$

Solving for y gives

$$\begin{aligned} y &\approx \frac{y_{n+2}y_n - y_{n+1}^2}{y_{n+2} - 2y_{n+1} + y_n} \\ &= \frac{y_n y_{n+2} - 2y_n y_{n+1} + y_n^2 - y_n^2 + 2y_n y_{n+1} - y_{n+1}^2}{y_{n+2} - 2y_{n+1} + y_n} \\ &= \frac{y_n(y_{n+2} - 2y_{n+1} + y_n) - (y_{n+1} - y_n)^2}{(y_{n+2} - y_{n+1}) - (y_{n+1} - y_n)} \\ &= y_n - \frac{(y_{n+1} - y_n)^2}{(y_{n+2} - y_{n+1}) - (y_{n+1} - y_n)}. \end{aligned}$$

Aitken's Δ^2 method

$$\hat{y}_n = y_n - \frac{(y_{n+1} - y_n)^2}{(y_{n+2} - y_{n+1}) - (y_{n+1} - y_n)}. \quad (3)$$

Example

The sequence $\{y_n = \cos(1/n)\}_{n=1}^{\infty}$ converges linearly to $y = 1$.

n	y_n	\hat{y}_n
1	0.54030	0.96178
2	0.87758	0.98213
3	0.94496	0.98979
4	0.96891	0.99342
5	0.98007	0.99541
6	0.98614	
7	0.98981	

- $\{\hat{y}_n\}_{n=1}^{\infty}$ converges more rapidly to $y = 1$ than $\{y_n\}_{n=1}^{\infty}$.



Definition

For a given sequence $\{y_n\}_{n=0}^{\infty}$, the forward difference Δy_n is defined by

$$\Delta y_n = y_{n+1} - y_n, \quad \text{for } n \geq 0.$$

Higher powers of Δ are defined recursively by

$$\Delta^k y_n = \Delta(\Delta^{k-1} y_n), \quad \text{for } k \geq 2.$$

The definition implies that

$$\Delta^2 y_n = \Delta(y_{n+1} - y_n) = \Delta y_{n+1} - \Delta y_n = (y_{n+2} - y_{n+1}) - (y_{n+1} - y_n).$$

So the formula for \hat{y}_n in (3) can be written as

$$\hat{y}_n = y_n - \frac{(\Delta y_n)^2}{\Delta^2 y_n}, \quad \text{for } n \geq 0.$$



Theorem

Suppose $\{y_n\}_{n=0}^{\infty} \rightarrow y$ linearly and

$$\lim_{n \rightarrow \infty} \frac{y_{n+1} - y}{y_n - y} < 1.$$

Then $\{\hat{y}_n\}_{n=0}^{\infty} \rightarrow y$ faster than $\{y_n\}_{n=0}^{\infty}$ in the sense that

$$\lim_{n \rightarrow \infty} \frac{\hat{y}_n - y}{y_n - y} = 0.$$

- Aitken's Δ^2 method constructs the terms in order:

$$y_0, \quad y_1 = g(y_0), \quad y_2 = g(y_1), \quad \hat{y}_0 = \{\Delta^2\}(y_0), \quad y_3 = g(y_2), \\ \hat{y}_1 = \{\Delta^2\}(y_1), \quad \dots$$

$$\Rightarrow \text{Assume } |\hat{y}_0 - y| < |y_2 - y|$$



- Steffensen's method constructs the terms in order:

$$\begin{aligned} y_0^{(0)} &\equiv y_0, & y_1^{(0)} &= g(y_0^{(0)}), & y_2^{(0)} &= g(y_1^{(0)}), \\ y_0^{(1)} &= \{\Delta^2\}(y_0^{(0)}), & y_1^{(1)} &= g(y_0^{(1)}), & y_2^{(1)} &= g(y_1^{(1)}), & \dots \end{aligned}$$

Steffensen's method (To find a solution of $y = g(y)$)

Given y_0 , tolerance TOL , maximum number of iteration M .

Set $i = 1$.

While $i \leq M$

Set $y_1 = g(y_0)$; $y_2 = g(y_1)$; $y = y_0 - (y_1 - y_0)^2 / (y_2 - 2y_1 + y_0)$.

If $|y - y_0| < Tol$, then STOP.

Set $i = i + 1$; $y_0 = y$.

End While

Theorem

Suppose that $x = g(x)$ has the solution x^ with $g'(x^*) \neq 1$. If $\exists \delta > 0$ such that $g \in C^3[x^* - \delta, x^* + \delta]$, then Steffensen's method gives quadratic convergence for any $x_0 \in [x^* - \delta, x^* + \delta]$.*

Zeros of polynomials and Müller's method

- Horner's method:

Let

$$\begin{aligned}P(x) &= a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n \\&= a_0 + x(a_1 + x(a_2 + \cdots + x(a_{n-1} + a_nx) \cdots))\end{aligned}$$

If

$$\begin{aligned}b_n &= a_n, \\b_k &= a_k + b_{k+1}x_0, \quad \text{for } k = n-1, n-2, \dots, 1, 0,\end{aligned}$$

then

$$b_0 = a_0 + b_1x_0 = a_0 + (a_1 + b_2x_0)x_0 = \cdots = P(x_0).$$

Consider

$$Q(x) = b_1 + b_2x + \cdots + b_nx^{n-1}.$$



Then

$$\begin{aligned} b_0 + (x - x_0)Q(x) &= b_0 + (x - x_0)(b_1 + b_2x + \cdots + b_nx^{n-1}) \\ &= (b_0 - b_1x_0) + (b_1 - b_2x_0)x + \cdots + (b_{n-1} - b_nx_0)x^{n-1} + b_nx^n \\ &= a_0 + a_1x + \cdots + a_nx^n = P(x). \end{aligned}$$

Differentiating $P(x)$ with respect to x gives

$$P'(x) = Q(x) + (x - x_0)Q'(x) \quad \text{and} \quad P'(x_0) = Q(x_0).$$

Use Newton-Raphson method to find an approximate zero of $P(x)$:

$$x_{k+1} = x_k - \frac{P(x_k)}{Q(x_k)}, \quad \forall k = 0, 1, 2, \dots$$

Similarly, let

$$\begin{aligned} c_n &= b_n = a_n, \\ c_k &= b_k + c_{k+1}x_k, \quad \text{for } k = n-1, n-2, \dots, 1, \end{aligned}$$

then $c_1 = Q(x_k)$.



Horner's method (Evaluate $y = P(x_0)$ and $z = P'(x_0)$)

Set $y = a_n; z = a_n$.

For $j = n - 1, n - 2, \dots, 1$

Set $y = a_j + yx_0; z = y + zx_0$.

End for

Set $y = a_0 + yx_0$.

If x_N is an approximate zero of P , then

$$\begin{aligned} P(x) &= (x - x_N)Q(x) + b_0 = (x - x_N)Q(x) + P(x_N) \\ &\approx (x - x_N)Q(x) \equiv (x - \hat{x}_1)Q_1(x). \end{aligned}$$

So $x - \hat{x}_1$ is an approximate factor of $P(x)$ and we can find a second approximate zero of P by applying Newton's method to $Q_1(x)$. The procedure is called deflation.

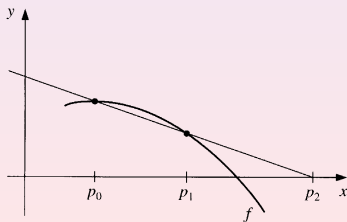


- Müller's method for complex root:

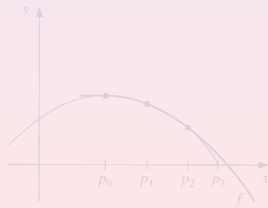
Theorem

If $z = a + ib$ is a complex zero of multiplicity m of $P(x)$ with real coefficients, then $\bar{z} = a - bi$ is also a zero of multiplicity m of $P(x)$ and $(x^2 - 2ax + a^2 + b^2)^m$ is a factor of $P(x)$.

Secant method: Given p_0 and p_1 , determine p_2 as the intersection of the x -axis with the line through $(p_0, f(p_0))$ and $(p_1, f(p_1))$.



Müller's method: Given p_0, p_1 and p_2 , determine p_3 by the intersection of the x -axis with the parabola through $(p_0, f(p_0))$, $(p_1, f(p_1))$ and $(p_2, f(p_2))$.

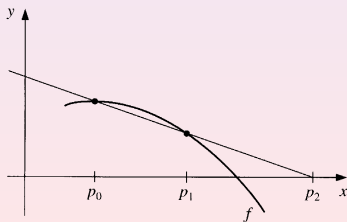


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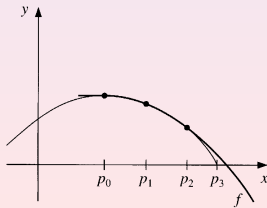
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Müller's method: Given p_0, p_1 and p_2 , determine p_3 by the intersection of the x -axis with the parabola through $(p_0, f(p_0))$, $(p_1, f(p_1))$ and $(p_2, f(p_2))$.



Let

$$P(x) = a(x - p_2)^2 + b(x - p_2) + c$$

that passes through $(p_0, f(p_0))$, $(p_1, f(p_1))$ and $(p_2, f(p_2))$. Then

$$f(p_0) = a(p_0 - p_2)^2 + b(p_0 - p_2) + c,$$

$$f(p_1) = a(p_1 - p_2)^2 + b(p_1 - p_2) + c,$$

$$f(p_2) = a(p_2 - p_2)^2 + b(p_2 - p_2) + c = c.$$

It implies that

$$c = f(p_2),$$

$$b = \frac{(p_0 - p_2)^2 [f(p_1) - f(p_2)] - (p_1 - p_2)^2 [f(p_0) - f(p_2)]}{(p_0 - p_2)(p_1 - p_2)(p_0 - p_1)},$$

$$a = \frac{(p_1 - p_2) [f(p_0) - f(p_2)] - (p_0 - p_2) [f(p_1) - f(p_2)]}{(p_0 - p_2)(p_1 - p_2)(p_0 - p_1)}.$$



To determine p_3 , a zero of P , we apply the quadratic formula to $P(x) = 0$ and get

$$p_3 - p_2 = \frac{2c}{b \pm \sqrt{b^2 - 4ac}}.$$

Choose

$$p_3 = p_2 + \frac{2c}{b + \operatorname{sgn}(b)\sqrt{b^2 - 4ac}}$$

such that the denominator will be largest and result in p_3 selected as the closest zero of P to p_2 .



Müller's method (Find a solution of $f(x) = 0$)

Given p_0, p_1, p_2 ; tolerance TOL ; maximum number of iterations M

Set $h_1 = p_1 - p_0$; $h_2 = p_2 - p_1$;

$$\delta_1 = (f(p_1) - f(p_0))/h_1; \delta_2 = (f(p_2) - f(p_1))/h_2;$$

$$d = (\delta_2 - \delta_1)/(h_2 + h_1); i = 3.$$

While $i \leq M$

$$\text{Set } b = \delta_2 + h_2 d; D = \sqrt{b^2 - 4f(p_2)d}.$$

If $|b - D| < |b + D|$, then set $E = b + D$ else set $E = b - D$.

$$\text{Set } h = -2f(p_2)/E; p = p_2 + h.$$

If $|h| < TOL$, then STOP.

$$\text{Set } p_0 = p_1; p_1 = p_2; p_2 = p; h_1 = p_1 - p_0; h_2 = p_2 - p_1;$$

$$\delta_1 = (f(p_1) - f(p_0))/h_1; \delta_2 = (f(p_2) - f(p_1))/h_2;$$

$$d = (\delta_2 - \delta_1)/(h_2 + h_1); i = i + 1.$$

End while

