Exercise Set 1.3, page 36

1. (a) $\frac{1}{1}+\frac{1}{4}\cdots+\frac{1}{100}=1.53;\quad \frac{1}{100}+\frac{1}{81}+\ldots+\frac{1}{1}=1.54.$

   The actual value is 1.549. Significant round-off error occurs much earlier in the first method.

   (b) The following algorithm will sum the series $\sum_{i=1}^{N} x_i$ in the reverse order.

   INPUT $N; x_1, x_2, \ldots, x_N$
   OUTPUT SUM

   STEP 1 Set $SUM = 0$
   STEP 2 For $j = 1, \ldots, N$ set $i = N - j + 1$
   $SUM = SUM + x_i$
   STEP 3 OUTPUT(SUM);
   STOP.

2.

	Approximation	Absolute Error	Relative Error
(a)	2.715	3.282×10^{-3}	1.207×10^{-3}
(b)	2.716	2.282×10^{-3}	8.394×10^{-4}
(c)	2.716	2.282×10^{-3}	8.394×10^{-4}
(d)	2.718	2.818×10^{-4}	1.037×10^{-4}

3. (a) 2000 terms (b) 20,000,000,000 terms

4. 4 terms

5. 3 terms

6. (a) $O\left(\frac{1}{n}\right)$ (b) $O\left(\frac{1}{n^2}\right)$ (c) $O\left(\frac{1}{n^2}\right)$ (d) $O\left(\frac{1}{n}\right)$

7. The rates of convergence are:

   (a) $O(h^2)$ (b) $O(h)$ (c) $O(h^2)$ (d) $O(h)$

8. (a) $n(n+1)/2$ multiplications; $(n+2)(n-1)/2$ additions.

   (b) $\sum_{i=1}^{n} a_i \left(\sum_{j=1}^{i} b_j\right)$ requires n multiplications; $(n+2)(n-1)/2$ additions.

9. The following algorithm computes $P(x_0)$ using nested arithmetic.

   INPUT $n, a_0, a_1, \ldots, a_n, x_0$
   OUTPUT $y = P(x_0)$

   STEP 1 Set $y = a_n$.
   STEP 2 For $i = n-1, n-2, \ldots, 0$ set $y = x_0 y + a_i$.
   STEP 3 OUTPUT (y);
   STOP.

10. The following algorithm uses the most effective formula for computing the roots of a quadratic equation.

INPUT A, B, C.
OUTPUT x_1, x_2.

STEP 1 If $A = 0$ then
if $B = 0$ then OUTPUT ('NO SOLUTIONS');
STOP.
else set $x_1 = -C/B$;
OUTPUT ('ONE SOLUTION',x_1);
STOP.

STEP 2 Set $D = B^2 - 4AC$.

STEP 3 If $D = 0$ then set $x_1 = -B/(2A)$;
OUTPUT ('MULTIPLE ROOTS', x_1);
STOP.

STEP 4 If $D < 0$ then set
$b = \sqrt{-D}/(2A)$;
$a = -B/(2A)$;
OUTPUT ('COMPLEX CONJUGATE ROOTS');
$x_1 = a + bi$;
$x_2 = a - bi$;
OUTPUT (x_1, x_2);
STOP.

STEP 5 If $B \geq 0$ then set
$d = B + \sqrt{D}$;
$x_1 = -2C/d$;
$x_2 = -d/(2A)$
else set
$d = -B + \sqrt{D}$;
$x_1 = d/(2A)$;
$x_2 = 2C/d$.

STEP 6 OUTPUT (x_1, x_2);
STOP.

11. The following algorithm produces the product $P = (x - x_0), \ldots, (x - x_n)$.

INPUT $n, x_0, x_1, \cdots, x_n, x$
OUTPUT P.

STEP 1 Set $P = x - x_0$;
$i = 1$.

STEP 2 While $P \neq 0$ and $i \leq n$ set
$P = P \cdot (x - x_i)$;
$i = i + 1$

STEP 3 OUTPUT (P);
STOP.

12. The following algorithm determines the number of terms needed to satisfy a given tolerance.

INPUT number x, tolerance TOL, maximum number of iterations M.
OUTPUT number N of terms or a message of failure.

STEP 1 Set $SUM = (1-2x)/\left(1-x+x^2\right)$;
$S = (1+2x)/\left(1+x+x^2\right)$;
$N = 2$.

STEP 2 While $N \leq M$ do Steps 3–5.

STEP 3 Set $j = 2^{N-1}$;
$y = x^j$
$t_1 = \frac{jy}{x}(1-2y)$;
$t_2 = y(y-1)+1$;
$SUM = SUM + t_1/t_2$.

STEP 4 If $|SUM - S| < TOL$ then
OUTPUT (N);
STOP.

STEP 5 Set $N = N+1$.

STEP 6 OUTPUT('Method failed');
STOP.

When $TOL = 10^{-6}$, we need to have $N \geq 4$.

13. (a) If $|\alpha_n - \alpha|/(1/n^p) \leq K$, then $|\alpha_n - \alpha| \leq K(1/n^p) \leq K(1/n^q)$ since $0 < q < p$. Thus, $|\alpha_n - \alpha|/(1/n^p) \leq K$ and $\{\alpha_n\}_{n=1}^{\infty} \to \alpha$ with rate of convergence $O(1/n^p)$.

(b)

n	$1/n$	$1/n^2$	$1/n^3$	$1/n^5$
5	0.2	0.04	0.008	0.0016
10	0.1	0.01	0.001	0.0001
50	0.02	0.0004	8×10^{-6}	1.6×10^{-7}
100	0.01	10^{-4}	10^{-6}	10^{-8}

The most rapid convergence rate is $O(1/n^4)$.

14. (a) If $F(h) = L + O\left(h^p\right)$, there is a constant $k > 0$ such that

$$|F(h) - L| \leq kh^p,$$

for sufficiently small $h > 0$. If $0 < q < p$ and $0 < h < 1$, then $h^q > h^p$. Thus, $kh^p < kh^q$, so

$$|F(h) - L| \leq kh^q \quad \text{and} \quad F(h) = L + O\left(h^q\right).$$

(b) For various powers of h we have the entries in the following table.

h	h^2	h^3	h^4
0.5	0.25	0.125	0.0625
0.1	0.01	0.001	0.0001
0.01	0.0001	0.00001	10^{-8}
0.001	10^{-6}	10^{-9}	10^{-12}

The most rapid convergence rate is $O\left(h^4\right)$.

15. Suppose that for sufficiently small $|x|$ we have positive constants k_1 and k_2 independent of x, for which

$$|F_1(x) - L_1| \leq K_1|x|^\alpha \quad \text{and} \quad |F_2(x) - L_2| \leq K_2|x|^\beta.$$

Let $c = \max(|c_1|, |c_2|, 1)$, $K = \max(K_1, K_2)$, and $\delta = \max(\alpha, \beta)$.

(a) We have

$$\begin{aligned} |F(x) - c_1L_1 - c_2L_2| &= |c_1(F_1(x) - L_1) + c_2(F_2(x) - L_2)| \\ &\leq |c_1|K_1|x|^\alpha + |c_2|K_2|x|^\beta \\ &\leq cK[|x|^\alpha + |x|^\beta] \\ &\leq cK|x|^\gamma[1 + |x|^{\delta-\gamma}] \\ &\leq \tilde{K}|x|^\gamma, \end{aligned}$$

for sufficiently small $|x|$ and some constant $\tilde{K}$. Thus, $F(x) = c_1L_1 + c_2L_2 + O(x^\gamma)$.

(b) We have

$$\begin{aligned} |G(x) - L_1 - L_2| &= |F_1(c_1x) + F_2(c_2x) - L_1 - L_2| \\ &\leq K_1|c_1x|^\alpha + K_2|c_2x|^\beta \\ &\leq Kc^\delta[|x|^\alpha + |x|^\beta] \\ &\leq Kc^\delta|x|^\gamma[1 + |x|^{\delta-\gamma}] \\ &\leq \tilde{K}|x|^\gamma, \end{aligned}$$

for sufficiently small $|x|$ and some constant $\tilde{K}$. Thus, $G(x) = L_1 + L_2 + O(x^\gamma)$.

16. Since $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = x$ and $x_{n+1} = 1 + \frac{1}{x_n}$, we have $x = 1 + \frac{1}{x}$. This implies that $x = \left(1 + \sqrt{5}\right)/2$. This number is called the *golden ratio.* It appears frequently in mathematics and the sciences.

17. (a) 354224848179261915075 (b) $0.3542248538 \times 10^{21}$

(c) The result in part (a) is computed using exact integer arithmetic, and the result in part (b) is computed using 10-digit rounding arithmetic.

(d) The result in part (a) required traversing a loop 98 times.

(e) The result is the same as the result in part (a).

18. (a) $n = 50$ (b) $n = 500$

Solutions of Equations of One Variable

Exercise Set 2.1, page 51

1. $p_3 = 0.625$

2. (a) $p_3 = -0.6875$ (b) $p_3 = 1.09375$

3. The Bisection method gives:

   (a) $p_7 = 0.5859$ (b) $p_8 = 3.002$ (c) $p_7 = 3.419$

4. The Bisection method gives:

   (a) $p_7 = -1.414$ (b) $p_8 = 1.414$ (c) $p_7 = 2.727$ (d) $p_7 = -0.7265$

5. The Bisection method gives:

   (a) $p_{17} = 0.641182$ (b) $p_{17} = 0.257530$

   (c) For the interval $[-3, -2]$, we have $p_{17} = -2.191307$, and for the interval $[-1, 0]$, we have $p_{17} = -0.798164$.

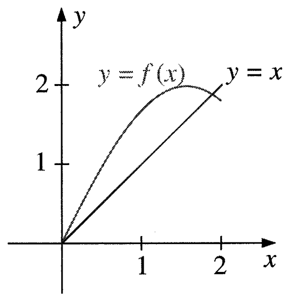
   (d) For the interval $[0.2, 0.3]$, we have $p_{14} = 0.297528$, and for the interval $[1.2, 1.3]$, we have $p_{14} = 1.256622$.

6. (a) $p_{17} = 1.51213837$ (b) $p_{17} = 0.97676849$

   (c) For the interval $[1, 2]$, we have $p_{17} = 1.41239166$, and for the interval $[2, 4]$, we have $p_{18} = 3.05710602$.

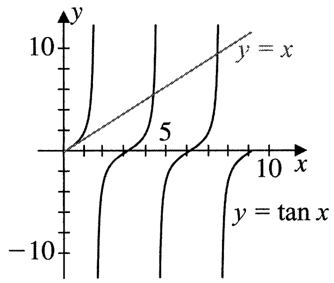
   (d) For the interval $[0, 0.5]$, we have $p_{16} = 0.20603180$, and for the interval $[0.5, 1]$, we have $p_{16} = 0.68196869$.

7. (a)



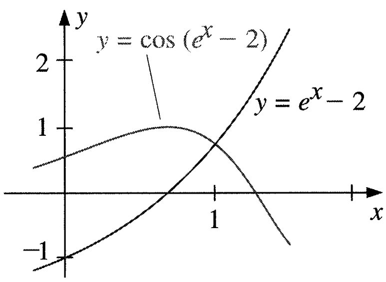
(b) Using $[1.5, 2]$ from part (a) gives $p_{16} = 1.89550018$.

8. (a)



(b) Using $[4.2, 4.6]$ from part (a) gives $p_{16} = 4.4934143$.

9. (a)



(b) $p_{17} = 1.00762177$

10. (a) 0 (b) 0 (c) 2 (d) -2

11. (a) 2 (b) -2 (c) -1 (d) 1

12. We have $\sqrt{3} \approx p_{14} = 1.7320$, using $[1, 2]$.

13. The third root of 25 is approximately $p_{14} = 2.92401$, using $[2, 3]$.

14. A bound for the number of iterations is $n \geq 12$ and $p_{12} = 1.3787$.

15. A bound is $n \geq 14$, and $p_{14} = 1.32477$.

16. For $n > 1$,

$$|f(p_n)| = \left(\frac{1}{n}\right)^{10} \leq \left(\frac{1}{2}\right)^{10} = \frac{1}{1024} < 10^{-3},$$

so

$$|p - p_n| = \frac{1}{n} < 10^{-3} \Leftrightarrow 1000 < n.$$

17. Since $\lim_{n\to\infty}(p_n - p_{n-1}) = \lim_{n\to\infty} 1/n = 0$, the difference in the terms goes to zero. However, p_n is the nth term of the divergent harmonic series, so $\lim_{n\to\infty} p_n = \infty$.

18. Since $-1 < a < 0$ and $2 < b < 3$, we have $1 < a + b < 3$ or $1/2 < 1/2(a + b) < 3/2$ in all cases. Further,

$$\begin{aligned} f(x) < 0, &\quad \text{for } -1 < x < 0 \quad \text{and} \quad 1 < x < 2; \\ f(x) > 0, &\quad \text{for } 0 < x < 1 \quad \text{and} \quad 2 < x < 3. \end{aligned}$$

Thus, $a_1 = a$, $f(a_1) < 0$, $b_1 = b$, and $f(b_1) > 0$.

(a) Since $a + b < 2$, we have $p_1 = \frac{a+b}{2}$ and $1/2 < p_1 < 1$. Thus, $f(p_1) > 0$. Hence, $a_2 = a_1 = a$ and $b_2 = p_1$. The only zero of f in $[a_2, b_2]$ is $p = 0$, so the convergence will be to 0.

(b) Since $a + b > 2$, we have $p_1 = \frac{a+b}{2}$ and $1 < p_1 < 3/2$. Thus, $f(p_1) < 0$. Hence, $a_2 = p_1$ and $b_2 = b_1 = b$. The only zero of f in $[a_2, b_2]$ is $p = 2$, so the convergence will be to 2.

(c) Since $a + b = 2$, we have $p_1 = \frac{a+b}{2} = 1$ and $f(p_1) = 0$. Thus, a zero of f has been found on the first iteration. The convergence is to $p = 1$.

19. The depth of the water is 0.838 ft.

20. The angle θ changes at the approximate rate $w = -0.317059$.