Hw16

2. (i) Show that if $p<-1$, and $c_i \in [x_{i-1}, x_i]$, then $h\sum_{i=2}^{n} c_i^p - \int_h^1 x^p dx = O(h^{p+1})$

Pf:

Since $p<-1 \Rightarrow x^p$ is decreasing on $(0,1]$

$$\Rightarrow x_i^p \le c_i^p \le x_{i-1}^p$$

$$\Rightarrow \int_{2h}^1 x^p dx + h\cdot 1 \le h\sum_{i=2}^{n} x_i^p \le h\sum_{i=2}^{n} c_i^p \le h\sum_{i=2}^{n} x_{i-1}^p \le \int_h^1 x^p dx + h\cdot h^p$$

$$\Rightarrow \underbrace{h - \frac{1}{p+1}\left((2h)^{p+1} - h^{p+1}\right)}_{O(h^{p+1})} \le h\sum_{i=2}^{n} c_i^p - \int_h^1 x^p dx \le h^{p+1}$$

$O(h^{p+1})$, since $p<-1 \Rightarrow p+1<0<1$

$\Rightarrow h^{p+1} > h$ as $h \to 0$

Hence, $h\sum_{i=2}^{n} c_i^p - \int_h^1 x^p dx = O(h^{p+1})$

(ii) $h\sum_{i=1}^{n} x_{i-\frac{1}{2}}^q - \int_0^1 x^q dx = O(h^{q+1})$, $-1<q<1$

Since $\int_{x_{i-1}}^{x_i} f(x)dx = hf(x_{i-\frac{1}{2}}) + \frac{h^3}{3} f''(\xi_i)$, where $\xi_i \in (x_{i-1}, x_i)$, $f(x)=x^q$

$$\Rightarrow h\sum_{i=1}^{n} x_{i-\frac{1}{2}}^q - \int_0^1 x^q dx = \left[h\cdot\left(\frac{h}{2}\right)^q - \int_0^h x^q dx\right] + \sum_{i=2}^{n} \frac{h^3}{3} f''(\xi_i)$$

Since $\deg(f'') = q-2 < -1 \Rightarrow h\sum_{i=2}^{n} f''(\xi_i) = q(q-1)\, h\sum_{i=2}^{n} \xi_i^{q-2}$

$$= q(q-1)\left[\int_h^1 x^{q-2} dx + O(h^{q-1})\right]$$

$$\Rightarrow h\sum_{i=1}^{n} x_{i-\frac{1}{2}}^q - \int_0^1 x^q dx = \left[2^{-q} h^{q+1} - \frac{1}{q+1} h^{q+1}\right] + \underbrace{\frac{h^2}{3} q\left[1 - h^{q-1} + (q-1)O(h^{q-1})\right]}_{O(h^{q+1})}$$

$O(h^{q+1})$, since $-1<q<1 \Rightarrow q+1<2$

$$= O(h^{q+1})$$

3. The order of convergence of Simpson's rule applied to $\int_{2h}^{1} x^p\,dx$, $-1<p<3$

Similar to 2 (ii)

$$\int_{x_{2(i-1)}}^{x_{2i}} f(x)\,dx = \frac{h}{3}[f(x_{2i-2})+4f(x_{2i-1})+f(x_{2i})] - \frac{h^5}{90} f^{(4)}(\xi_i), \text{ where } \xi_i \in (x_{2(i-1)}, x_{2i}),\ f(x)=x^p$$

Let $S_n = \frac{h}{3}\left[f(x_0)+4\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{n/2-1} f(x_{2i}) + f(x_n)\right]$

$$\Rightarrow S_{n-2} - \int_{2h}^{1} x^p\,dx = \sum_{i=2}^{n/2}\left(-\frac{h^5}{90}f^{(4)}(\xi_i)\right) = -\frac{h^4}{180}\left[2h\sum_{i=2}^{n/2} f^{(4)}(\xi_i)\right]$$

Since $\deg(f^{(4)}) = p-4 < -1 \Rightarrow 2h\sum_{i=2}^{n/2} f^{(4)}(\xi_i) = p(p-1)(p-2)(p-3)\,2h\sum_{i=2}^{n/2}\xi_i^{p-4}$

$$= p(p-1)(p-2)(p-3)\left[\int_{2h}^{1} x^{p-4}\,dx + O(h^{p-3})\right]$$

(a) $-1<p<0$

$$S_{n-2} - \int_{2h}^{1} x^p\,dx = \underbrace{-\frac{h^4}{180}p(p-1)(p-2)\left[1-(2h)^{p-3}+O(h^{p-3})\right]}_{=\,O(h^{p+1}),\ \text{since } -1<p<3 \Rightarrow p+1<4}$$

(b) $0<p<1$

$$S_n - \int_{0}^{1} x^p\,dx = \frac{h}{3}[f(x_0)+4f(x_1)+f(x_2)] + S_{n-2} - \int_{0}^{2h} x^p\,dx - \int_{2h}^{1} x^p\,dx$$

$$= \underbrace{\frac{h}{3}[4h^p+(2h)^p] - \frac{1}{p+1}(2h)^{p+1}}_{=\,O(h^{p+1})} + \underbrace{\left(S_{n-2}-\int_{2h}^{1} x^p\,dx\right)}_{=\,O(h^{p+1})}$$

$$= O(h^{p+1})$$