(e) 7.103932, $2+\frac{1}{2}\pi^2$ (f) 1.428074, $\frac{1}{2}(e^2+1)-e$

16. Gaussian quadrature with $n=m=p=3$ gives:

(a) 5.206442 (b) 0.08333333 (c) 0.07166667

(d) 0.08333333 (e) 6.928161 (f) 1.474577

17. Algorithm 4.6 with $n=m=p=4$ gives the first listed value. The second is from Algorithm 4.6 with $n=m=p=5$.

(a) 5.206447, 5.206447 (b) 0.08333333,0.08333333 (c) 0.07142857,0.07142857

(d) 0.08333333,0.08333333 (e) 6.934912,6.934801 (f) 1.476207, 1.476246

18. Gaussian quadrature with $n=m=p=4$ gives 3.0521250. The exact result is 3.0521249.
19. The approximation 20.41887 requires 125 functional evaluations.

Exercise Set 4.9, page 245

1. The Composite Simpson's rule gives:

(a) 0.5284163 (b) 4.266654 (c) 0.4329748 (d) 0.8802210

2. The Composite Simpson's Rule gives:

(a) 1.076163 (b) 20.07458

3. The Composite Simpson's rule gives:

(a) 0.4112649 (b) 0.2440679 (c) 0.05501681 (d) 0.2903746

4. The Composite Simpson's Rule gives:

(a) 1.1107218 with $n=16$ (b) 0.58904782 with $n=12$

5. The escape velocity is approximately 6.9450 mi/s.

6. The polynomial $L_n(x)$ has n distinct zeros in $[0,\infty)$. Let $x_1, ..., x_n$ be the n distinct zeros of L_n and define, for each $i = 1, ..., n$,

$$c_{n,i} = \int_0^\infty e^{-x} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}\, dx.$$

Let $P(x)$ be any polynomial of degree $n-1$ or less, and let $P_{n-1}(x)$ be the $(n-1)$th Lagrange polynomial for P on the nodes $x_1, ..., x_n$. As in the proof of Theorem 4.7,

$$\int_0^\infty P(x)e^{-x}\, dx = \int_0^\infty P_{n-1}(x)e^{-x}\, dx = \sum_{i=1}^{n} c_{n,i}P(x_i),$$

so the quadrature formula is exact for polynomials of degree $n-1$ or less.
If $P(x)$ has degree $2n-1$ or less, then $P(x)$ can be divided by the nth Laguerre polynomial $L_n(x)$ to obtain

$$P(x) = Q(x)L_n(x) + R(x),$$

where $Q(x)$ and $R(x)$ are both polynomials of degree less than n. As in proof of Theorem 4.7, the orthogonality of the Laguerre polynomials on $[0,\infty)$ implies that

$$Q(x) = \sum_{i=0}^{n-1} d_i L_i(x),$$

for some constants d_i.

Thus,

$$\begin{aligned}\int_0^\infty e^{-x}P(x)\, dx &= \int_0^\infty \sum_{i=0}^{n-1} d_i L_i(x)L_n(x)e^{-x}\, dx + \int_0^\infty e^{-x}R(x)\, dx\\ &= \sum_{i=0}^{n-1} d_i \int_0^\infty L_i(x)L_n(x)e^{-x}\, dx + \sum_{i=1}^{n} c_{n,i}R(x_i)\\ &= 0 + \sum_{i=1}^{n} c_{n,i}R(x_i) = \sum_{i=1}^{n} c_{n,i}R(x_i).\end{aligned}$$

But,

$$P(x_i) = Q(x_i)L_n(x_i) + R(x_i) = 0 + R(x_i) = R(x_i),$$

so

$$\int_0^\infty e^{-x}P(x)\, dx = \sum_{i=1}^{n} c_{n,i}P(x_i).$$

Hence the quadrature formula has degree of precision $2n-1$.

7. (a) $\displaystyle\int_0^\infty e^{-x}f(x)\,dx \approx 0.8535534\ f(0.5857864) + 0.1464466\ f(3.4142136)$

   (b) $\displaystyle\int_0^\infty e^{-x}f(x)\,dx \approx 0.7110930\ f(0.4157746) + 0.2785177\ f(2.2942804) + 0.0103893\ f(6.2899451)$

8. For $n = 2$ we have 0.9238795. For $n = 3$ we have 0.9064405.

9. For $n = 2$ we have 2.9865139. For $n = 3$ we have 2.9958198.

Direct Methods for Solving Linear Systems

Exercise Set 6.1, page 356

1. (a) Intersecting lines with solution $x_1 = x_2 = 1$.
   (b) One line, so there is an infinite number of solutions with $x_2 = \frac{3}{2} - \frac{1}{2}x_1$.
   (c) One line, so there is an infinite number of solutions with $x_2 = -\frac{1}{2}x_1$.
   (d) Intersecting lines with solution $x_1 = \frac{2}{7}$ and $x_2 = -\frac{11}{7}$.

2. (a) Intersecting lines whose solution is $x_1 = x_2 = 0$.
   (b) Parallel lines, so there is no solution.
   (c) Three lines in the plane that do not intersect at a common point.
   (d) Two planes in space which intersect in a line with $x_1 = -\frac{5}{4}x_2$ and $x_3 = \frac{3}{2}x_2 + 1$.

3. Gaussian elimination gives the following solutions.

   (a) $x_1 = 1.0,\ x_2 = -0.98,\ x_3 = 2.9$ (b) $x_1 = 1.1,\ x_2 = -1.1,\ x_3 = 2.9$

4. Gaussian elimination gives the following solutions.

   (a) $x_1 = -0.70, x_2 = 1.1, x_3 = 2.9$ (b) $x_1 = -0.88, x_2 = 0.74, x_3 = 3.0$

5. Gaussian elimination gives the following solutions.
   (a) $x_1 = 1.1875, x_2 = 1.8125, x_3 = 0.875$ with one row interchange required
   (b) $x_1 = -1, x_2 = 0, x_3 = 1$ with no interchange required
   (c) $x_1 = 1.5, x_2 = 2, x_3 = -1.2, x_4 = 3$ with no interchange required
   (d) No unique solution

6. Gaussian elimination gives the following solutions.
   (a) $x_1 = -4,\ x_2 = -8,\ x_3 = -6$ with one row interchange required
   (b) $x_1 = \frac{22}{9},\ x_2 = -\frac{4}{9},\ x_3 = \frac{4}{3}, x_4 = 1$ with one row interchange required
   (c) $x_1 = 13,\ x_2 = 8,\ x_3 = 8, x_4 = 5$ with one row interchange required.
   (d) $x_1 = -1,\ x_2 = 2,\ x_3 = 0, x_4 = 1$ with one row interchange required.

7. Gaussian elimination with `DIGITS:=10` gives the following solutions:

   (a) $x_1 = -227.0769$, $x_2 = 476.9231$, $x_3 = -177.6923$;
   (b) $x_1 = 1.001291$, $x_2 = 1$, $x_3 = 1.00155$;
   (c) $x_1 = -0.03174600$, $x_2 = 0.5952377$, $x_3 = -2.380951$, $x_4 = 2.777777$;
   (d) $x_1 = 1.918129$, $x_2 = 1.964912$, $x_3 = -0.9883041$, $x_4 = -3.192982$,$x_5 = -1.134503$.

8. Gaussian elimination with `DIGITS:=10` gives the following solutions:

   (a) $x_1 = 0.9798657720$, $x_2 = 4.281879191$, $x_3 = 17.48322147$;
   (b) $x_1 = 6.461447620$, $x_2 = 8.394321092$, $x_3 = -0.01347368618$;
   (c) $x_1 = 1.349448559$, $x_2 = -4.67798776$, $x_3 = -4.032893779$, $x_4 = -1.656637732$;
   (d) $x_1 = 13.49999998$, $x_2 = -11.5000000000$, $x_3 = 23.75000003$, $x_4 = 121.5000003$,$x_5 = 97.75000025$.

9. (a) When $\alpha = -1/3$, there is no solution.
   (b) When $\alpha = 1/3$, there is an infinite number of solutions with $x_1 = x_2 + 1.5$, and x_2 is arbitrary.
   (c) If $\alpha \neq \pm 1/3$, then the unique solution is

$$x_1 = \frac{3}{2(1+3\alpha)} \quad \text{and} \quad x_2 = \frac{-3}{2(1+3\alpha)}.$$

10. (a) $\alpha = 1$ (b) $\alpha = -1$

    (c) $x_1 = -1/(1-\alpha), x_2 = 1, x_3 = 1/(1-\alpha)$

11. Suppose $x_1', ..., x_n'$ is a solution to the linear system (6.1).
(i) The new system becomes

$$\begin{aligned} E_1 &: a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ &\vdots \\ E_i &: \lambda a_{i1}x_1 + \lambda a_{i2}x_2 + ... + \lambda a_{in}x_n = \lambda b_i \\ &\vdots \\ E_n &: a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n. \end{aligned}$$

Clearly, $x_1', ..., x_n'$ satisfies this system. Conversely, if $x_1^*, ..., x_n^*$ satisfies the new system, dividing E_i by λ shows $x_1^*, ..., x_n^*$ also satisfies (6.1).
(ii) The new system becomes

$$\begin{aligned} E_1 &: a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ &\vdots \\ E_i &: (a_{i1} + \lambda a_{j1})x_1 + (a_{i2} + \lambda a_{j2})x_2 + ... + (a_{in} + \lambda a_{jn})x_n = b_i + \lambda b_j \\ &\vdots \\ E_n &: a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n. \end{aligned}$$

Clearly, $x'_1, ..., x'_n$ satisfies all but possibly the ith equation. Multiplying E_j by λ gives

$$\lambda a_{j1}x'_1 + \lambda a_{j2}x'_2 + ... + \lambda a_{jn}x'_n = \lambda b_j,$$

which can be subtracted from E_i in the new system results in the system (6.1). Thus, $x'_1, ..., x'_n$ satisfies the new system. Conversely, if $x^*_1, ..., x^*_n$ is a solution to the new system, then all but possibly E_i of (6.1) are satisfied by $x^*_1, ..., x^*_n$. Multiplying E_j of the new system by $-\lambda$ gives

$$-\lambda a_{j1}x^*_1 - \lambda a_{j2}x^*_2 - ... - \lambda a_{jn}x^*_n = -\lambda b_j.$$

Adding this to E_i in the new system produces E_i of (6.1). Thus, $x^*_1, ..., x^*_n$ is a solution of (6.1).
(iii) The new system and the old system have the same set of equations to satisfy. Thus, they have the same solution set.

12. Change Algorithm 6.1 as follows:

    STEP 1 For $i = 1, \ldots, n$ do STEPS 2, 3, and 4.

    STEP 4 For $j = 1, \ldots, i-1, i+1, \ldots, n$ do STEPS 5 and 6.

    STEP 8 For $i = 1, \ldots, n$ set $x_i = a_{i,n+1}/a_{ii}$.

    In addition, delete STEP 9.

13. The Gauss-Jordan method gives the following results.

    (a) $x_1 = 0.98, x_2 = -0.98, x_3 = 2.9$

    (b) $x_1 = 1.1, x_2 = -1.0, x_3 = 2.9$

14. The Gauss-Jordan method with single precision arithmetic gives the following solutions.

    (a) $x_1 = -227.0787,\ x_2 = 476.9262,\ x_3 = -177.6934$

    (b) $x_1 = 1.000036,\ x_2 = 0.9999991,\ x_3 = 0.9986052$

    (c) $x_1 = -0.03177120,\ x_2 = 0.5955572,\ x_3 = -2.381768,\ x_4 = 2.778329$

    (d) $x_1 = 1.918129,\ x_2 = 1.964912,\ x_3 = -0.9883036,\ x_4 = -3.192982,\ x_5 = -1.134503$

15. The results for are listed in the following table. (The abbreviations M/D and A/S are used for multiplications/divisions and additions/subtractions, respectively.)

	Gaussian elimination		Gauss-Jordan	
n	M/D	A/S	M/D	A/S
3	17	11	21	12
10	430	375	595	495
50	44150	42875	64975	62475
100	343300	338250	509950	499950

16. (a) The Gaussian elimination procedure requires

$$\frac{(2n^3 + 3n^2 - 5n)}{6} \text{ Multiplications/Divisions}$$

and

$$\frac{n^3 - n}{3} \text{ Additions/Subtractions.}$$

The additional elimination steps are:

$$\text{For } i = n, n-1, ..., 2$$

$$\text{for } j = 1, ..., i-1,$$

$$\text{set} \quad a_{j,n+1} = a_{j,n+1} - \frac{a_{ji}a_{i,n+1}}{a_{ii}}.$$

This requires

$$n(n-1) \text{ Multiplications/Divisions}$$

and

$$\frac{n(n-1)}{2} \text{ Additions/Subtractions.}$$

Solving for

$$x_i = \frac{a_{i,n+1}}{a_{ii}}$$

requires n divisions. Thus, the totals are

$$\frac{n^3}{3} + \frac{3n^2}{2} - \frac{5n}{6} \text{ Multiplications/Divisions}$$

and

$$\frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6} \text{ Additions/Subtractions.}$$

(b) The results are listed in the following table. In this table the abbreviations M/D and A/S are used for Multiplications/Divisions and for Additions/Subtractions, respectively.

	Gaussian Elimination		Gauss-Jordan		Hybrid	
n	M/D	A/S	M/D	A/S	M/D	A/S
3	17	11	21	12	20	11
10	430	375	595	495	475	375
50	44150	42875	64975	62475	45375	42875
100	343300	338250	509950	499950	348250	338250

17. The Gaussian-Elimination–Gauss-Jordan hybrid method gives the following results.

(a) $x_1 = 1.0, x_2 = -0.98, x_3 = 2.9$ (b) $x_1 = 1.0, x_2 = -1.0, x_3 = 2.9$

18. The Gauss-Jordan hybrid method with single-precision arithmetic gives the following solutions.

(a) -227.0788, 476.9262, -177.6934 (b) 0.9990999, 0.9999991, 0.9986052

(c) -0.03177060, 0.5955554, -2.381768, 2.778329

(d) $x_1 = 1.918126$, $x_2 = 1.964916$, $x_3 = -0.9883027$, $x_4 = -3.192982$, $x_5 = -1.134503$

19. (a) There is sufficient food to satisfy the average daily consumption.

(b) We could add 200 of species 1, or 150 of species 2, or 100 of species 3, or 100 of species 4.

(c) Assuming none of the increases indicated in part (b) was selected, species 2 could be increased by 650, or species 3 could be increased by 150, or species 4 could be increased by 150.

(d) Assuming none of the increases indicated in parts (b) or (c) were selected, species 3 could be increased by 150, or species 4 could be increased by 150.

20. (a) For the Trapezoidal rule $m = n = 1$, $x_0 = 0$, $x_1 = 1$ so that for $i = 0$ and 1, we have

$$\begin{aligned} u(x_i) &= f(x_i) + \int_0^1 K(x_i,t)u(t)\ dt \\ &= f(x_i) + \frac{1}{2}\left[K(x_i,0)u(0) + K(x_i,1)u(1)\right]. \end{aligned}$$

Substituting for x_i gives the desired equations.

(b) We have $n = 4$, $h = \frac{1}{4}$, $x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{2}$, $x_3 = \frac{3}{4}$, and $x_4 = 1$, so

$$\begin{aligned} u(x_i) =& f(x_i) + \frac{h}{2}\Bigg[K(x_i,0)u(0) + 2K\left(x_i,\frac{1}{4}\right)u\left(\frac{1}{4}\right) \\ &+ 2K\left(x_i,\frac{1}{2}\right)u\left(\frac{1}{2}\right) + 2K\left(x_i,\frac{3}{4}\right)u\left(\frac{3}{4}\right) + K(x_i,1)u(1)\Bigg], \end{aligned}$$

for $i = 0, 1, 2, 3, 4$. This gives

$$u(x_i) = x_i^2 + \frac{1}{8}\left[e^{x_i}u(0) + 2e^{|x_i-\frac{1}{4}|}u\left(\frac{1}{4}\right) + 2e^{|x_i-\frac{1}{2}|}u\left(\frac{1}{2}\right) + 2e^{|x_i-\frac{3}{4}|}u\left(\frac{3}{4}\right) + e^{|x_i-1|}u(1)\right],$$

for each $i = 1, \ldots, 4$. The 5×5 linear system has solution $u(0) = -1.154255$, $u\left(\frac{1}{4}\right) = -0.9093298$, $u\left(\frac{1}{2}\right) = -0.7153145$, $u\left(\frac{3}{4}\right) = -0.5472949$, and $u(1) = -0.3931261$.

(c) The Composite Simpson's rule gives

$$\begin{aligned} \int_0^1 K(x_i,t)u(t)\ dt =& \frac{h}{3}\Bigg[K(x_i,0)u(0) + 4K\left(x_i,\frac{1}{4}\right)u\left(\frac{1}{4}\right) + 2K\left(x_i,\frac{1}{2}\right)u\left(\frac{1}{2}\right) + \\ & 4K\left(x_i,\frac{3}{4}\right)u\left(\frac{3}{4}\right) + K(x_i,1)u(1)\Bigg], \end{aligned}$$

which results in the linear equations

$$u(x_i) = x_i^2 + \frac{1}{12}\left[e^{x_i}u(0) + 4e^{|x_i-\frac{1}{4}|}u\left(\frac{1}{4}\right) + 2e^{|x_i-\frac{1}{2}|}u\left(\frac{1}{2}\right) + 4e^{|x_i-\frac{3}{4}|}u\left(\frac{3}{4}\right) + e^{|x_i-1|}u(1)\right].$$

The 5×5 linear system has solutions $u(0) = -1.234286$, $u\left(\frac{1}{4}\right) = -0.9507292$, $u\left(\frac{1}{2}\right) = -0.7659400$, $u\left(\frac{3}{4}\right) = -0.5844737$, and $u(1) = -0.4484975$.

Exercise Set 6.2, page 368

1. The following row interchanges are required for these systems.

   (a) none

   (b) Interchange rows 2 and 3.

   (c) none

   (d) Interchange rows 1 and 2.

2. The following row interchanges are required for these systems.

   (a) none

   (b) none

   (c) none

   (d) none

3. The following row interchanges are required for these systems.

   (a) Interchange rows 1 and 2.

   (b) Interchange rows 1 and 3.

   (c) Interchange rows 1 and 2, then interchange rows 2 and 3.

   (d) Interchange rows 1 and 2.

4. The following row interchanges are required for these systems.

   (a) Interchange rows 2 and 3.

   (b) Interchange rows 1 and 3.

   (c) Interchange rows 1 and 3, then interchange rows 2 and 3.

   (d) Interchange rows 1 and 2.

5. The following row interchanges are required for these systems.

   (a) Interchange rows 1 and 3, then interchange rows 2 and 3.

   (b) Interchange rows 2 and 3.

   (c) Interchange rows 2 and 3.

   (d) Interchange rows 1 and 3, then interchange rows 2 and 3.