

Hw/3

26(a) To solve the equation

$$\int_0^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = 0.45$$

We can use Newton's method to find the root of

$$f(x) = \int_0^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt - 0.45$$

$$\Rightarrow f'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \text{ decreasing as } x \rightarrow \infty$$

In each iteration, we need to calculate $f(x_k)$ and hence to

calculate $\int_0^{x_k} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$. Here we use Simpson's rule to get a suitable

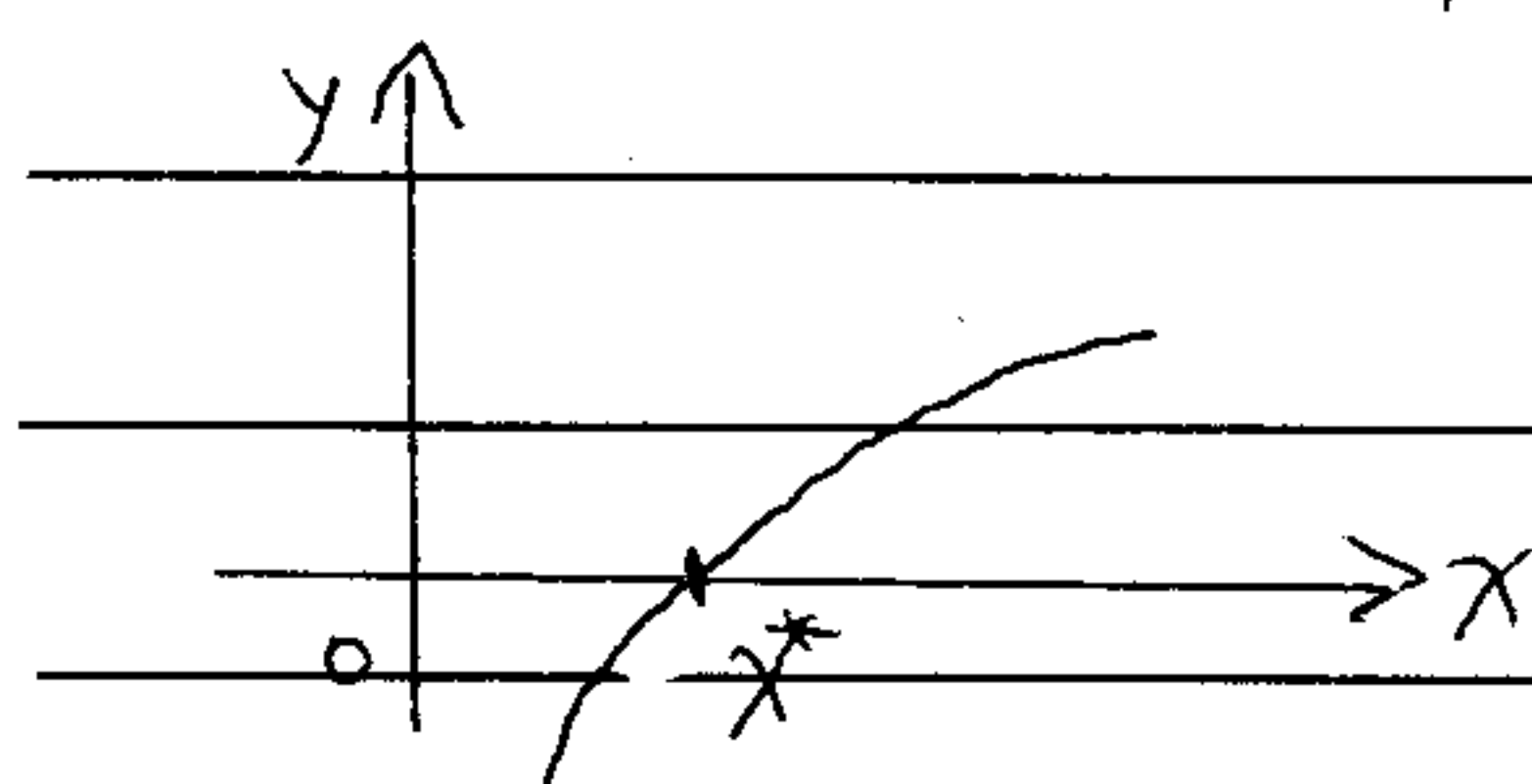
approximation of the integration.

Let x^* be the root of $f(x)$; by Mean Value Theorem

$$\frac{f(x_k) - f(x^*)}{x_k - x^*} = f'(\xi_k) \text{ where } \xi_k \text{ is between } x^* \text{ \& } x_k$$

Since $f' > 0, \forall x$ and $f''(x) = -\frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} < 0, \forall x > 0$, the graph of

$f(x)$ near x^* would be \Rightarrow



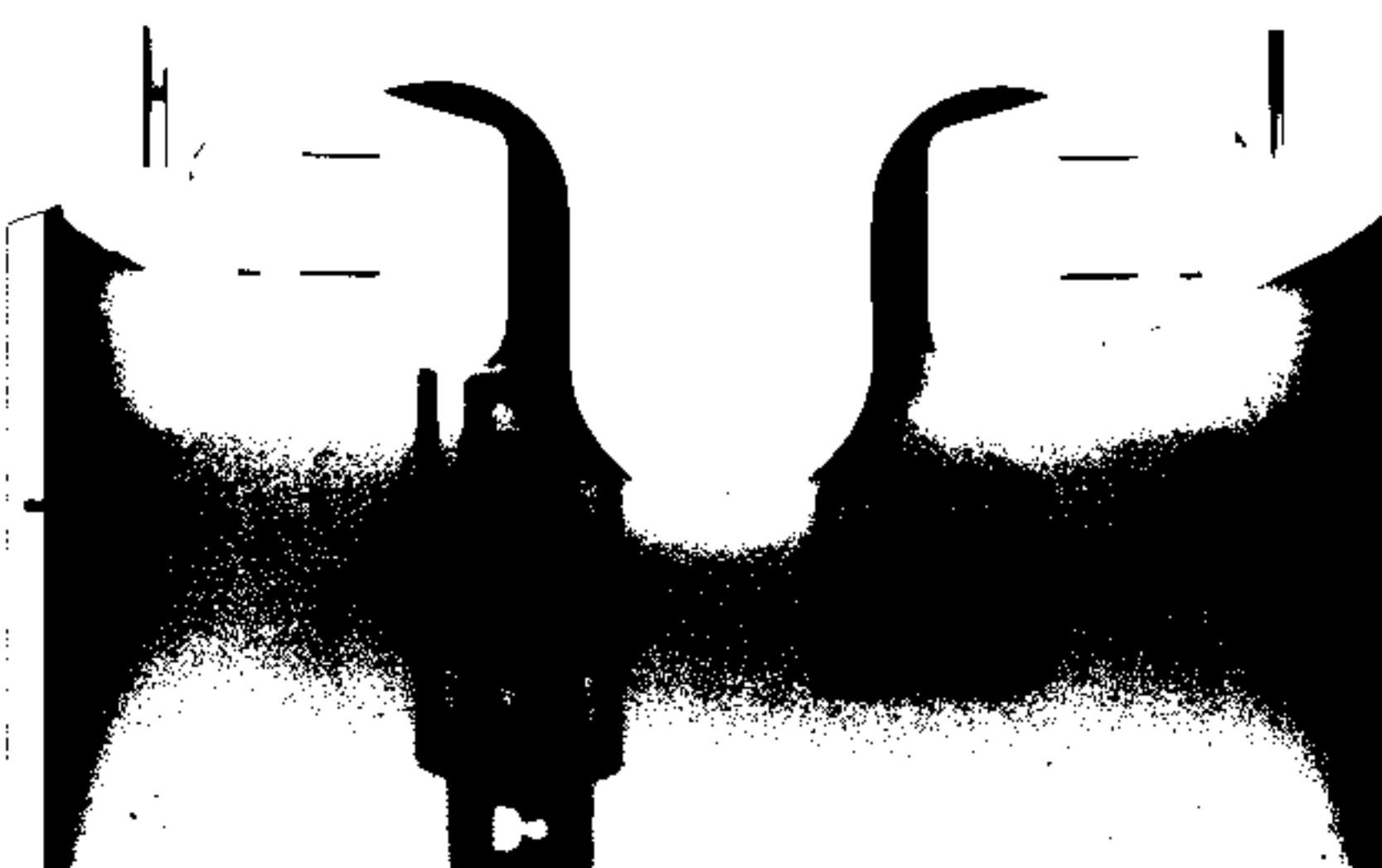
(Increasing and concave down)

Hence we may assume that our Newton's iteration would

not be too far away from x^* as x_k approaches to x^* .

So we can estimate a lower bound for $f'(\xi_k)$.

Per-Duet



Using Simpson's rule to calculate $\int_0^3 \frac{1}{\sqrt{\pi}} e^{-\frac{t^2}{2}} dt$ with $|\text{error}| < 10^{-4}$

$$\text{Since } f^{(4)}(x) = -\frac{1}{\sqrt{\pi}} x(x^2-3)e^{-\frac{x^2}{2}} \text{ and } f^{(5)}(x) = \frac{1}{\sqrt{\pi}} (x^4-6x^2+3)e^{-\frac{x^2}{2}}$$

$$f^{(5)}(x) = 0 \Rightarrow x^2 = 3 \pm \sqrt{6} \Rightarrow x = \pm \sqrt{3 \pm \sqrt{6}}$$

$$\Rightarrow \max_{x \in \mathbb{R}} f^{(4)}(x) \doteq 0.551$$

$$\left| -\frac{3-0}{180} \left(\frac{3-0}{n}\right)^4 f^{(4)}(\mu) \right| < 10^{-4} \Rightarrow n \geq 9.2851$$

$$\text{Choose } n=10 \Rightarrow \int_0^3 \frac{1}{\sqrt{\pi}} e^{-\frac{t^2}{2}} dt \approx 0.49864 > 0.45 \Rightarrow x^* \in [0, 3]$$

$$\Rightarrow f'(3_k) \geq f'(3) \doteq 4.4318 \times 10^{-3}$$

$$\Rightarrow \frac{f(x_k) - f(x^*)}{x_k - x^*} = f'(3_k) \geq f'(3)$$

$$\Rightarrow |x_k - x^*| \leq \frac{|f(x_k)|}{f'(3)} \leq \frac{1}{f'(3)} (|f_h(x_k)| + |\text{error bound}|)$$

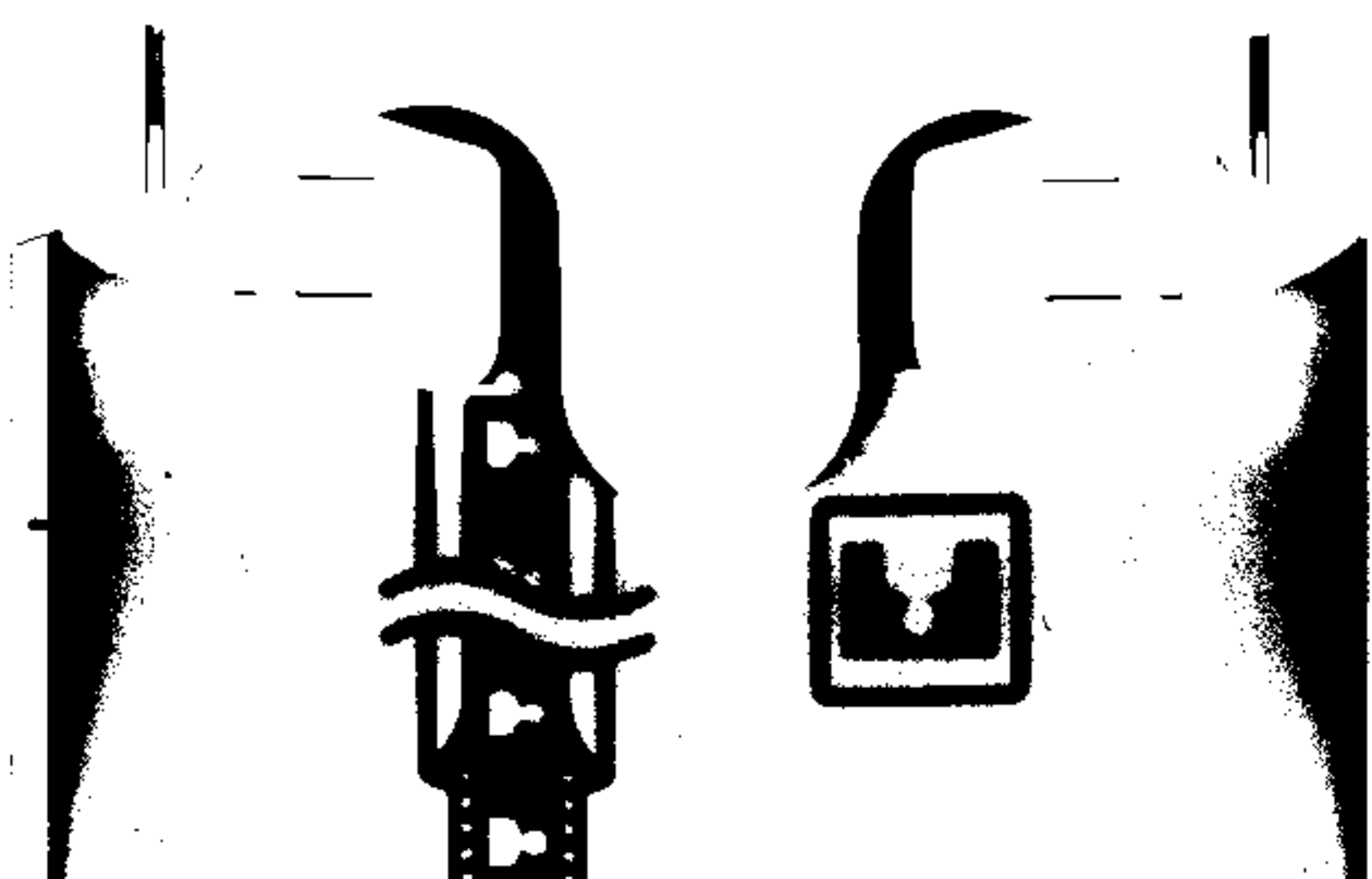
Hence we can estimate $|x_k - x^*|$ by the numerical integration $f_h(x_k)$ and error bound with Simpson's rule.

In fact, the error bound changes with x_k , so we have to estimate an upper bound for the error.

$$\text{Since } x^* \in [0, 3], |\text{error bound}| \leq \left| -\frac{3-0}{180} \left(\frac{3-0}{n}\right)^4 f^{(4)}(\xi) \right| \text{ where } \xi \in (0, 3)$$

and we want to find x_k such that $|x_k - x^*| < 10^{-5}$.

Hence we have to choose n such that



$$\frac{1}{f(3)} \left| \frac{3}{180} \cdot \left(\frac{3}{n}\right)^4 \cdot \max_{z \in (0,3)} f^{(4)}(z) \right| \leq \frac{1}{2} \times 10^{-5} \Rightarrow n \geq 63.99 \dots$$

\Rightarrow choose $n=64$

and check whether $\frac{|f_h(x_k)|}{f(3)} \leq \frac{1}{2} \times 10^{-5}$ or not.

If x_k satisfies the condition \uparrow , by the inequality

$$|x_k - x^*| \leq \frac{1}{f(3)} (|f_h(x_k)| + |\text{error bound}|) \leq 10^{-5}$$

We can make sure x_k is what we want

Numerical result:

$$\text{choose } n=64, p_0=0.5 \Rightarrow p_5 = 1.644853624494629$$