(b) We have

$$P_{0,2}(x) = \frac{\left(x - h^4\right) N_2\left(\frac{h}{2}\right)}{\frac{h^4}{16} - h^4} + \frac{\left(x - \frac{h^4}{16}\right) N_2(h)}{h^4 - \frac{h^4}{16}}, \quad \text{so} \quad P_{0,2}(0) = \frac{16N_2\left(\frac{h}{2}\right) - N_2(h)}{15}.$$

14. All the approximations of the form $N_{2i}\left(h/2^j\right)$, for $i = 1, 2, \ldots$ and $j = 0, 1, 2, \ldots$, will be upper bounds for M, and all the approximations of the form $N_{2i+1}\left(\frac{h}{2^j}\right)$, for $i = 0, 1, 2, \ldots$ and $j = 0, 1, 2, \ldots$, will be lower bounds for M.

15. (a) The polygonal approximations are in the following table.

k	4	8	16	32	64	128	256	512
p_k	$2\sqrt{2}$	3.0614675	3.1214452	3.1365485	3.1403312	3.1412723	3.1415138	3.1415729
P_k	4	3.3137085	3.1825979	3.1517249	3.144184	3.1422236	3.1417504	3.1416321

(b) Values of p_k and P_k are given in the following tables, together with the extrapolation results:

For p_k we have :

2.8284271				
3.0614675	3.1391476			
3.1214452	3.1414377	3.1415904		
3.1365485	3.1415829	3.1415926	3.1415927	
3.1403312	3.1415921	3.1415927	3.1415927	3.1415927

For P_k we have :

4				
3.3137085	3.0849447			
3.1825979	3.1388943	3.1424910		
3.1517249	3.1414339	3.1416032	3.1415891	
3.1441184	3.1415829	3.1415928	3.1415926	3.1415927

Exercise Set 4.3, page 195

1. The Trapezoidal rule gives the following approximations.

(a) 0.265625 (b) −0.2678571 (c) −0.17776434 (d) 0.1839397

(e) −0.8666667 (f) −0.1777643 (g) 0.2180895 (h) 4.1432597

2. The Trapezoidal rule gives the following approximations.

(a) 0.4693956405 (b) 0.08664339760 (c) −0.03702425262 (d) 0.2863341726

3. For the approximations in Exercise 1 we have the following.

	Actual error	Error bound
(a)	0.071875	0.125
(b)	7.943×10^{-4}	9.718×10^{-4}
(c)	0.0358147	0.0396972
(d)	0.0233369	0.1666667
(e)	0.1326975	0.5617284
(f)	9.443×10^{-4}	1.0707×10^{-3}
(g)	0.0663431	0.0807455
(h)	1.554631	2.298827

4. For the approximations in Exercise 2 we have the following.

	Actual error	Error bound
(a)	0.0203171288	0.02083333333
(b)	0.03407359031	0.0625
(c)	0.01664745664	0.02444080544
(d)	0.0138202920	0.02904245657

5. Simpson's rule gives the following approximations.

(a) 0.1940104 (b) −0.2670635 (c) 0.1922453 (d) 0.16240168

(e) −0.7391053 (f) −0.1768216 (g) 0.1513826 (h) 2.5836964

6. Simpson's rule gives the following approximations.

(a) 0.4897985467 (b) 0.05285463857 (c) −0.02027158961 (d) 0.2762704525

7. Simpson's rule gives the following approximations.

	Actual error	Error bound
(a)	2.604×10^{-4}	2.6042×10^{-4}
(b)	7.14×10^{-7}	9.92×10^{-7}
(c)	1.406×10^{-5}	2.170×10^{-5}
(d)	1.7989×10^{-3}	4.1667×10^{-4}
(e)	5.1361×10^{-3}	0.063280
(f)	1.549×10^{-6}	2.095×10^{-6}
(g)	3.6381×10^{-4}	4.1507×10^{-4}
(h)	4.9322×10^{-3}	0.1302826

8.

	Actual error	Error bound
(a)	0.0000857774	0.0000868056
(b)	0.00028483128	0.001215277778
(c)	0.00010520637	0.0001147849363
(d)	0.0001565719	0.0005334208049

9. The Midpoint rule gives the following approximations.

(a) 0.1582031 (b) −0.2666667 (c) 0.1743309 (d) 0.1516327

(e) −0.6753247 (f) −0.1768200 (g) 0.1180292 (h) 1.8039148

10. The Midpoint rule gives the following approximations.

(a) 0.5 (b) 0.03596025906 (c) −0.01189525810 (d) 0.2658385924

11. The Midpoint rule gives the following approximations.

	Actual error	Error bound
(a)	0.0355469	0.0625
(b)	3.961×10^{-4}	4.859×10^{-4}
(c)	0.0179285	0.0198486
(d)	8.9701×10^{-3}	0.0833333
(e)	0.0564448	0.2808642
(f)	4.698×10^{-4}	5.353×10^{-4}
(g)	0.0337172	0.0403728
(h)	0.7847138	1.1494136

12.

	Actual error	Error bound
(a)	0.0102872307	0.01041666667
(b)	0.01660954823	0.03125
(c)	0.00848153788	0.01222040272
(d)	0.0066752882	0.01452122828

13. $f(1) = \frac{1}{2}$

14. Simpson's rule gives the result $\frac{13}{3}$.

15. The degree of precision is 3.

16. The degree of precision is 3.

17. $c_0 = \frac{1}{3}$, $c_1 = \frac{4}{3}$, $c_2 = \frac{1}{3}$

18. $c_0 = \frac{7}{3}$, $c_1 = -\frac{2}{3}$, $c_2 = \frac{1}{3}$

19. $c_0 = c_1 = \frac{1}{2}$ gives the highest degree of precision, which is 2.

20. $c_1 = \frac{1}{2}$, $x_0 = 0.211324865$ and $x_1 = 0.788675135$ give the highest degree of precision, 3.

21. The following approximations are obtained from Formula (4.23) through Formula (4.30), respectively.

    (a) 0.1024404, 0.1024598, 0.1024598, 0.1024598, 0.1024695, 0.1024663, 0.1024598, and 0.1024598
    (b) 0.7853982, 0.7853982, 0.7853982, 0.7853982, 0.7853982, 0.7853982, 0.7853982, and 0.7853982
    (c) 1.497171, 1.477536, 1.477529, 1.477523, 1.467719, 1.470981, 1.477512, and 1.477515
    (d) 4.950000, 2.740909, 2.563393, 2.385700, 1.636364, 1.767857, 2.074893, and 2.116379
    (e) 3.293182, 2.407901, 2.359772, 2.314751, 1.965260, 2.048634, 2.233251, and 2.249001
    (f) 0.5000000, 0.6958004, 0.7126032, 0.7306341, 0.7937005, 0.7834709, 0.7611137, and 0.7593572

22.

i	t_i	w_i	$y(t_i)$	
(4.23)	(4.24)	(4.26)	(4.27)	(4.29)
5.43476	5.03420	5.03292	4.83393	5.03180

23. The errors in Exercise 16 are 1.6×10^{-6}, 5.3×10^{-8}, -6.7×10^{-7}, -7.2×10^{-7}, and -1.3×10^{-6}, respectively.

24. For

$$\begin{aligned}
f(x) =& x : a_0x_0 + a_1(x_0+h) + a_2(x_0+2h) = 2x_0h + 2h^2;\\
f(x) =& x^2 : a_0x_0^2 + a_1(x_0+h)^2 + a_2(x_0+2h)^2 = 2x_0^2h + 4x_0h^2 + \frac{8h^3}{3};\\
f(x) =& x^3 : a_0x_0^3 + a_1(x_0+h)^3 + a_2(x_0+2h)^3 = 2x_0^3h + 6x_0^2h^2 + 8x_0h^3 + 4h^4.
\end{aligned}$$

Solving this linear system for a_0, a_1, and a_2 gives $a_0 = \frac{h}{3}, a_1 = \frac{4h}{3}$, and $a_2 = \frac{h}{3}$. Using $f(x) = x^4$ gives $f^{(4)}(\xi) = 24$, so

$$\frac{1}{5}\left(x_2^5 - x_0^5\right) = \frac{h}{3}\left(x_0^4 + 4x_1^4 + x_2^4\right) + 24k.$$

Replacing x_1 with $x_0 + h$, x_2 with $x_0 + 2h$ and simplifying gives $k = -h^5/90$.

25. If $E(x^k) = 0$, for all $k = 0, 1, \ldots, n$ and $E(x^{n+1}) \neq 0$, then with $p_{n+1}(x) = x^{n+1}$, we have a polynomial of degree $n+1$ for which $E(p_{n+1}(x)) \neq 0$. Let $p(x) = a_nx^n + \cdots + a_1x + a_0$ be any polynomial of degree less than or equal to n. Then $E(p(x)) = a_nE(x^n) + \cdots + a_1E(x) + a_0E(1) = 0$. Conversely, if $E(p(x)) = 0$, for all polynomials of degree less than or equal to n, it follows that $E(x^k) = 0$, for all $k = 0, 1, \ldots, n$. Let $p_{n+1}(x) = a_{n+1}x^{n+1} + \cdots + a_0$ be a polynomial of degree $n+1$ for which $E(p_{n+1}(x)) \neq 0$. Since $a_{n+1} \neq 0$, we have

$$x^{n+1} = \frac{1}{a_{n+1}}p_{n+1}(x) - \frac{a_n}{a_{n+1}}x^n - \cdots - \frac{a_0}{a_{n+1}}.$$

Then

$$\begin{aligned}
E(x^{n+1}) &= \frac{1}{a_{n+1}}E(p_{n+1}(x)) - \frac{a_n}{a_{n+1}}E(x^n) - \cdots - \frac{a_0}{a_{n+1}}E(1)\\
&= \frac{1}{a_{n+1}}E(p_{n+1}(x)) \neq 0.
\end{aligned}$$

Thus, the quadrature formula has degree of precision n.

26. Using $n = 3$ in Theorem 4.2 gives

$$\int_a^b f(x)dx = \sum_{i=0}^{3} a_if(x_i) + \frac{h^5f^{(4)}(\xi)}{24}\int_0^3 t(t-1)(t-2)(t-3)dt.$$

Since

$$\int_0^3 t(t-1)(t-2)(t-3)dt = -\frac{9}{10},$$

the error term is

$$-3h^5 f^{(4)}(\xi)/80.$$

Also,

$$a_i = \int_{x_0}^{x_3} \prod_{\substack{j=0 \\ j\neq i}}^{3} \frac{x - x_j}{x_i - x_j} dx, \quad \text{for each} \quad i = 0, 1, 2, 3.$$

Using the change of variables $x = x_0 + th$ gives

$$a_i = h \int_0^3 \prod_{\substack{j=0 \\ j\neq i}}^{3} \frac{t - j}{i - j} dt, \quad \text{for each} \quad i = 0, 1, 2, 3.$$

Evaluating the integrals gives $a_0 = \frac{3h}{8}, a_1 = \frac{9h}{8}, a_2 = \frac{9h}{8}$, and $a_3 = \frac{3h}{8}$.

Exercise Set 4.4, page 203

1. The Composite Trapezoidal rule approximations are:

(a) 0.639900 (b) 31.3653 (c) 0.784241 (d) −6.42872

(e) −13.5760 (f) 0.476977 (g) 0.605498 (h) 0.970926

2.

	Composite Trapezoidal Approximation	Actual Integral
(a)	0.91193343	0.92073549
(b)	0.09363001	0.08802039
(c)	−0.66468785	−0.66293045
(d)	0.36487225	0.36423547

3. The Composite Simpson's rule approximations are:

(a) 0.99999998 (b) 1.9999999 (c) 2.2751458 (d) −19.646796

4.

	Composite Simpson's Approximation	Actual Integral
(a)	0.92088605	0.92073549
(b)	0.08809221	0.08802039
(c)	−0.66292308	−0.66293045
(d)	0.36423967	0.36423547