Numerical Differentiation and Integration

Exercise Set 4.1, page 176

1. From the forward-backward difference formula (4.1), we have the following approximations:

   (a) $f'(0.5) \approx 0.8520$, $f'(0.6) \approx 0.8520$, $f'(0.7) \approx 0.7960$

   (b) $f'(0.0) \approx 3.7070$, $f'(0.2) \approx 3.1520$, $f'(0.4) \approx 3.1520$

2. The approximations are in the following tables

(a)

x	$f(x)$	$f'(x)$
-0.3	1.9507	0.9140
-0.2	2.0421	0.9140
-0.1	2.0601	0.1800

(b)

x	$f(x)$	$f'(x)$
1.0	1.0000	1.3125
1.2	1.2625	1.3125
1.4	1.6595	1.9850

3. The approximations are in the following tables.

(a)

x	Actual Error	Error Bound
0.5	0.0255	0.0282
0.6	0.0267	0.0282
0.7	0.0312	0.0322

(b)

x	Actual Error	Error Bound
0.0	0.2930	0.3000
0.2	0.2694	0.2779
0.4	0.2602	0.2779

4. (a)

x	Actual Error	Error Bound
-0.3	0.34457	0.36842
-0.2	0.35633	0.36842
-0.1	0.38533	0.39203

(b)

x	Actual Error	Error Bound
1.0	0.31250	0.33646
1.2	0.32507	0.33646
1.4	0.35712	0.36729

5. For the endpoints of the tables, we use Formula (4.4). The other approximations come from Formula (4.5).

   (a) $f'(1.1) \approx 17.769705$, $f'(1.2) \approx 22.193635$, $f'(1.3) \approx 27.107350$, $f'(1.4) \approx 32.150850$

   (b) $f'(8.1) \approx 3.092050$, $f'(8.3) \approx 3.116150$, $f'(8.5) \approx 3.139975$, $f'(8.7) \approx 3.163525$

   (c) $f'(2.9) \approx 5.101375$, $f'(3.0) \approx 6.654785$, $f'(3.1) \approx 8.216330$, $f'(3.2) \approx 9.786010$

   (d) $f'(2.0) \approx 0.13533150$, $f'(2.1) \approx -0.09989550$, $f'(2.2) \approx -0.3298960$, $f'(2.3) \approx -0.5546700$

6. For the endpoints of the tables, we use Formula (4.4). The other approximations come from Formula (4.5).

(a)

x	$f(x)$	$f'(x)$
-0.3	-0.27652	-0.06030
-0.2	-0.25074	0.57590
-0.1	-0.16134	1.25370
0.0	0.0	1.97310

(b)

x	$f(x)$	$f'(x)$
7.4	-68.3193	-16.6933
7.6	-71.6982	-17.0958
7.8	-75.1576	-17.4980
8.0	-78.6974	-17.9000

(c)

x	$f(x)$	$f'(x)$
1.1	1.52918	1.34360
1.2	1.64024	0.87760
1.3	1.70470	0.36265
1.4	1.71277	-0.20125

(d)

x	$f(x)$	$f'(x)$
-2.7	0.054797	-0.915178
-2.5	0.11342	1.50141
-2.3	0.65536	2.17825
-2.1	0.98472	1.11535

7. The errors and error bounds are given in the following tables.

(a)

x	Actual Error	Error Bound
1.1	0.280322	0.359033
1.2	0.147282	0.179517
1.3	0.179874	0.219262
1.4	0.378444	0.438524

(b)

x	Actual Error	Error Bound
8.1	0.00018594	0.000020322
8.3	0.00010551	0.000010161
8.5	9.116×10^{-5}	0.000009677
8.7	0.00020197	0.000019355

(c)

x	Actual Error	Error Bound
2.9	0.011956	0.0180988
3.0	0.0049251	0.00904938
3.1	0.0004765	0.00493920
3.2	0.0013745	0.00987840

(d)

x	Actual Error	Error Bound
2.0	0.00252235	0.00410304
2.1	0.00142882	0.00205152
2.2	0.00204851	0.00260034
2.3	0.00437954	0.00520068

8. (a)

x	Actual Error	Error Bound
−0.3	0.028638	0.029692
−0.2	0.014097	0.014846
−0.1	0.013577	0.014130
0.0	0.026900	0.028260

(b)

x	Actual Error	Error Bound
7.4	0.000367	0.000032
7.6	0.000083	0.000016
7.8	0.000041	0.000015
8.0	0.000000	0.000030

(c)

x	Actual Error	Error Bound
1.1	0.033886	0.034784
1.2	0.016791	0.017392
1.3	0.015740	0.016817
1.4	0.030920	0.033633

(d)

x	Actual Error	Error Bound
−2.7	0.511122	1.440958
−2.5	0.435980	0.720479
−2.3	0.632733	0.720479
−2.1	1.044472	1.440958

9. The approximations and the formulas used are:

(a) $f'(2.1) \approx 3.899344$ from (4.7) $f'(2.2) \approx 2.876876$ from (4.7) $f'(2.3) \approx 2.249704$ from (4.6) $f'(2.4) \approx 1.837756$ from (4.6) $f'(2.5) \approx 1.544210$ from (4.7) $f'(2.6) \approx 1.355496$ from (4.7)

(b) $f'(-3.0) \approx -5.877358$ from (4.7) $f'(-2.8) \approx -5.468933$ from (4.7) $f'(-2.6) \approx -5.059884$ from (4.6) $f'(-2.4) \approx -4.650223$ from (4.6) $f'(-2.2) \approx -4.239911$ from (4.7) $f'(-2.0) \approx -3.828853$ from (4.7)

10. The approximations are in the following tables.

(a)

x	$f(x)$	$f'(x)$
1.05	−1.709847	7.798690
1.10	−1.373823	5.753747
1.15	−1.119214	4.499409
1.20	−0.9160143	3.675512
1.25	−0.7470223	3.088414
1.30	−0.6015966	2.710997

(b)

x	$f(x)$	$f'(x)$
− 3.0	16.08554	−19.08087
−2.8	12.64465	−15.44088
−2.6	9.863738	−12.46303
−2.4	7.623176	−10.02259
−2.2	5.825013	−8.02097
−2.0	4.389056	−6.38573

11. The approximations are in the following tables.

(a)

x	Actual Error	Error Bound
2.1	0.0242312	0.109271
2.2	0.0105138	0.0386885
2.3	0.0029352	0.0182120
2.4	0.0013262	0.00644808
2.5	0.0138323	0.109271
2.6	0.0064225	0.0386885

(b)

x	Actual Error	Error Bound
− 3.0	1.55×10^{-5}	6.33×10^{-7}
−2.8	1.32×10^{-5}	6.76×10^{-7}
−2.6	7.95×10^{-7}	1.05×10^{-7}
−2.4	6.79×10^{-7}	1.13×10^{-7}
−2.2	1.28×10^{-5}	6.76×10^{-7}
−2.0	7.96×10^{-6}	6.76×10^{-7}

12. (a)

x	Actual Error	Error Bound
1.05	0.0484600	0.2185438
1.10	0.0210325	0.0773769
1.15	0.0058693	0.0364240
1.20	0.0026524	0.0128962
1.25	0.0276704	0.2185438
1.30	0.0128401	0.0773769

(b)

x	Actual Error	Error Bound
− 3.0	0.004666	0.006427
−2.8	0.003763	0.005262
−2.6	0.000711	0.001071
−2.4	0.000591	0.000877
−2.2	0.004041	0.006427
−2.0	0.003329	0.005262

13. $f'(3) \approx \frac{1}{12}[f(1) - 8f(2) + 8f(4) - f(5)] = 0.21062$, with an error bound given by

$$\max_{1 \le x \le 5} \frac{|f^{(5)}(x)|h^4}{30} \le \frac{23}{30} = 0.7\overline{6}.$$

14. $f'(3) \approx \frac{1}{2}[f(4) - f(2)] = 0.21210$, with an error bound given by

$$\max_{1\le x\le 5} \frac{|f'''(x)|\,h^2}{6} \le \frac{4}{2} = 0.6\bar{6}.$$

15. From the forward-backward difference formula (4.1), we have the following approximations:

(a) $f'(0.5) \approx 0.852$, $f'(0.6) \approx 0.852$, $f'(0.7) \approx 0.7960$

(b) $f'(0.0) \approx 3.707$, $f'(0.2) \approx 3.153$, $f'(0.4) \approx 3.153$

16. For the endpoints of the tables, we use Formula (4.7). The other approximations come from Formula (4.6).

(a) $f'(1.1) \approx 17.75$, $f'(1.2) \approx 22.17$, $f'(1.3) \approx 27.10$, $f'(1.4) \approx 32.50$,

(b) $f'(8.1) \approx 3.075$, $f'(8.3) \approx 3.125$, $f'(8.5) \approx 3.150$, $f'(8.7) \approx 3.150$,

(c) $f'(2.9) \approx 5.080$, $f'(3.0) \approx 6.655$, $f'(3.1) \approx 8.220$, $f'(3.2) \approx 9.760$,

(d) $f'(2.0) \approx 0.1600$, $f'(2.1) \approx -0.1000$, $f'(2.2) \approx -0.3300$, $f'(2.3) \approx -0.5500$,

17. For the endpoints of the tables, we use Formula (4.7). The other approximations come from Formula (4.6).

(a) $f'(2.1) \approx 3.884$ $f'(2.2) \approx 2.896$ $f'(2.3) \approx 2.249$ $f'(2.4) \approx 1.836$ $f'(2.5) \approx 1.550$ $f'(2.6) \approx 1.348$

(b) $f'(-3.0) \approx -5.883$ $f'(-2.8) \approx -5.467$ $f'(-2.6) \approx -5.059$ $f'(-2.4) \approx -4.650$ $f'(-2.2) \approx -4.208$ $f'(-2.0) \approx -3.875$

18. (a)

		$f'(0.4)$			$f''(0.4)$
(4.1)	$h = 0.6$	-0.8889958	(4.8)	$h = 0.2$	-1.191050
	$h = 0.4$	-0.6979043			
	$h = 0.2$	-0.5486810			
	$h = -0.2$	-0.3104710			
(4.4)	$h = 0.2$	-0.3994578			
(4.5)	$h = 0.2$	-0.4295760			

(b)

		$f'(0.4)$			$f''(0.4)$
(4.1)	$h = 0.4$	-1.059153	(4.8)	$h = 0.4$	-1.573943
	$h = 0.2$	-0.8471275		$h = 0.2$	-1.492233
	$h = -0.2$	-0.5486810			
	$h = -0.4$	-0.4295760			
(4.4)	$h = 0.2$	-0.6351018			
	$h = -0.2$	-0.6677860			
(4.5)	$h = 0.4$	-0.7443646			
	$h = 0.2$	-0.6979043			
(4.6)	$h = 0.2$	-0.6824175			

19. The approximation is -4.8×10^{-9}. $f''(0.5) = 0$. The error bound is 0.35874. The method is very accurate since the function is symmetric about $x = 0.5$.

20. With $h = 0.1$, we have 36.641, and with $h = 0.01$, we have 36.5. The actual value is 36.5935.

21. (a) $f'(0.2) \approx -0.1951027$ (b) $f'(1.0) \approx -1.541415$ (c) $f'(0.6) \approx -0.6824175$

22. We have the Taylor expansions:

$$
\begin{aligned}
f(x_0 - h) &= f(x_0) - hf'(x_0) + \frac{1}{2}h^2 f''(x_0) - \frac{1}{6}h^3 f'''(x_0) + \frac{1}{24}h^4 f^{(4)}(x_0) + O\left(h^5\right);\\
f(x_0 + h) &= f(x_0) + hf'(x_0) + \frac{1}{2}h^2 f''(x_0) + \frac{1}{6}h^3 f'''(x_0) + \frac{1}{24}h^4 f^{(4)}(x_0) + O\left(h^5\right);\\
f(x_0 + 2h) &= f(x_0) + 2hf'(x_0) + 2h^2 f''(x_0) + \frac{4}{3}h^3 f'''(x_0) + \frac{2}{3}h^4 f^{(4)}(x_0) + O\left(h^5\right);\\
f(x_0 + 3h) &= f(x_0) + 3hf'(x_0) + \frac{9}{2}h^2 f''(x_0) + \frac{9}{2}h^3 f'''(x_0) + \frac{27}{8}h^4 f^{(4)}(x_0) + O\left(h^5\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&Af(x_0 - h) + Bf(x_0 + h) + Cf(x_0 + 2h) + Df(x_0 + 3h) = \\
&\quad f(x_0)(A + B + C + D) + f'(x_0)h[-A + B + 2C + 3D] + f''(x_0)h^2\left(\frac{1}{2}A + \frac{1}{2}B + 2C + \frac{9}{2}D\right)\\
&\quad + f'''(x_0)h^3\left(-\frac{1}{6}A + \frac{1}{6}B + \frac{4}{3}C + \frac{9}{2}D\right) + f^{(4)}(x_0)h^4\left(\frac{1}{24}A + \frac{1}{24}B + \frac{2}{3}C + \frac{27}{8}D\right).
\end{aligned}
$$

We want to eliminate the terms involving $f''(x_0)$, $f'''(x_0)$, and $f^{(4)}(x_0)$ and have the coefficient of $f'(x_0)$ equal 1. Thus,

$$-A+B+2C+3D=1$$
$$\frac{1}{2}A+\frac{1}{2}B+2C+\frac{9}{2}D=0$$
$$-\frac{1}{6}A+\frac{1}{6}B+\frac{4}{3}C+\frac{9}{2}D=0$$
$$\frac{1}{24}A+\frac{1}{24}B+\frac{2}{3}C+\frac{27}{8}D=0.$$

The solution to this linear system is

$$A=-\frac{1}{4},\quad B=\frac{3}{2},\quad C=-\frac{1}{2},\quad \text{and}\quad D=\frac{1}{12}.$$

Thus,

$$-\frac{1}{4}f(x_0-h)+\frac{3}{2}f(x_0+h)-\frac{1}{2}f(x_0+2h)+\frac{1}{12}f(x_0+3h)=f(x_0)\left(-\frac{1}{4}+\frac{3}{2}-\frac{1}{2}+\frac{1}{12}\right)+hf'(x_0)+O\left(h^5\right).$$

Solving for $f'(x_0)$ gives

$$f'(x_0)=-\frac{1}{h}\left[f(x_0)\frac{10}{12}+\frac{1}{4}f(x_0-h)-\frac{3}{2}f(x_0+h)+\frac{1}{2}f(x_0+2h)-\frac{1}{12}f(x_0+3h)\right]+O\left(h^4\right).$$

Finally,

$$f'(x_0)=\frac{1}{12h}\left[-3f(x_0-h)-10f(x_0)+18f(x_0+h)-6f(x_0+2h)+f(x_0+3h)\right]+O\left(h^4\right).$$

23. $f'(0.4)\approx -0.4249840$ and $f'(0.8)\approx -1.032772$.

24. (a) Assume that the computed values $\tilde{f}(x_0+h)$ and $\tilde{f}(x_0)$ are related to the true values $f(x_0+h)$ and $f(x_0)$ by the formulas $f(x_0+h)=\tilde{f}(x_0+h)+e(x_0+h)$ and $f(x_0)=\tilde{f}(x_0)+e(x_0)$. The total error in the approximation becomes

$$f'(x_0)-\frac{\tilde{f}(x_0+h)-\tilde{f}(x_0)}{h}=\frac{e(x_0+h)-e(x_0)}{h}-\frac{h}{2}f''(\xi_0).$$

If $|e(x_0+h)|<\varepsilon$, $|e(x_0)|<\varepsilon$, and $|f''(\xi_0)|\le M$, then

$$\left|f'(x_0)-\frac{\tilde{f}(x_0+h)-\tilde{f}(x_0)}{h}\right|\le\frac{2\varepsilon}{h}+\frac{hM}{2}.$$

(b) The function in Example 2 is

$$f(x)=xe^x,\quad \text{for}\quad 1.8\le x\le 2.2.$$

We have $f'(x)=xe^x+e^x$ and $f''(x)=xe^x+2e^x$. Thus,

$$M=\max_{1.8\le x\le 2.2}|f''(x)|=f''(2.2)=37.9050567.$$

The numbers in the table are given to 6 decimal places, so it is reasonable to let $\varepsilon = 0.0000005$. The optimal value of h is

$$h = 2\sqrt{\frac{\varepsilon}{M}} = 2\sqrt{\frac{0.0000005}{37.9050567}} = 0.000229703.$$

25. The three-point formulas give the results in the following table.

Time	0	3	5	8	10	13
Speed	79	82.4	74.2	76.8	69.4	71.2

26. The three-point formulas give the results in the following table.

t	1.00	1.01	1.02	1.03	1.04
$\varepsilon(t)$	2.400	2.403	3.386	5.352	7.320

27. The approximations eventually become zero since the numerator becomes zero.

28. By averaging the Taylor polynomials we have

$$f'''(x_0) = \frac{1}{h^3}\left[-\frac{1}{2}f(x_0-2h) + f(x_0-h) - f(x_0+h) + \frac{1}{2}f(x_0+2h)\right] - \frac{h^2}{4}f^{(5)}(\xi),$$

where ξ is between $x_0 - 2h$ and $x_0 + 2h$.

29. Since $e'(h) = -\varepsilon/h^2 + hM/3$, we have $e'(h) = 0$ if and only if $h = \sqrt[3]{3\varepsilon/M}$. Also, $e'(h) < 0$ if $h < \sqrt[3]{3\varepsilon/M}$ and $e'(h) > 0$ if $h > \sqrt[3]{3\varepsilon/M}$, so an absolute minimum for $e(h)$ occurs at $h = \sqrt[3]{3\varepsilon/M}$.

Exercise Set 4.2, page 184

1. (a) $f'(1) \approx 1.0000109$ (b) $f'(0) \approx 2.0000000$

   (c) $f'(1.05) \approx 2.2751459$ (d) $f'(2.3) \approx -19.646799$

2. (a) $f'(1) \approx 0.99999998$ (b) $f'(0) \approx 1.9999999$

   (c) $f'(1.05) \approx 2.2751458$ (d) $f'(2.3) \approx -19.646796$

3. (a) $f'(1) \approx 1.001$ (b) $f'(0) \approx 1.999$

(c) $f'(1.05) \approx 2.283$ (d) $f'(2.3) \approx -19.61$

4. (a) $f'(1) \approx 0.9999$ (b) $f'(0) \approx 1.997$

(c) $f'(1.05) \approx 2.282$ (d) $f'(2.3) \approx -19.66$

5. $\int_0^\pi \sin x \, dx \approx 1.999999$
6. $\int_0^{3\pi/2} \cos x \, dx \approx -1.000135$
7. With $h = 0.1$, Formula (4.6) becomes

$$f'(2) \approx \frac{1}{1.2}\left[1.8e^{1.8} - 8\left(1.9e^{1.9}\right) + 8(2.1)e^{2.1} - 2.2e^{2.2}\right] = 22.166995.$$

With $h = 0.05$, Formula (4.6) becomes

$$f'(2) \approx \frac{1}{0.6}\left[1.9e^{1.9} - 8\left(1.95e^{1.95}\right) + 8(2.05)e^{2.05} - 2.1e^{2.1}\right] = 22.167157.$$

8. The formula $f'(x_0) = \frac{1}{12h}\left[f(x_0+4h) - 12f(x_0+2h) + 32f(x_0+h) - 21f(x_0)\right]$ is $O\left(h^3\right)$.
9. Let

$$N_2(h) = N\left(\frac{h}{3}\right) + \left(\frac{N\left(\frac{h}{3}\right) - N(h)}{2}\right) \quad \text{and} \quad N_3(h) = N_2\left(\frac{h}{3}\right) + \left(\frac{N_2\left(\frac{h}{3}\right) - N_2(h)}{8}\right).$$

Then $N_3(h)$ is an $O(h^3)$ approximation to M.

10. Let $N_2(h) = N\left(\frac{h}{3}\right) + \frac{1}{8}\left(N\left(\frac{h}{3}\right) - N(h)\right)$ and $N_3(h) = N_2\left(\frac{h}{3}\right) + \frac{1}{80}\left(N_2\left(\frac{h}{3}\right) - N_2(h)\right)$. Then $N_3(h)$ is an $O\left(h^6\right)$ approximation to M.
11. Let $N(h) = (1+h)^{1/h}$, $N_2(h) = 2N\left(\frac{h}{2}\right) - N(h)$, $N_3(h) = N_2\left(\frac{h}{2}\right) + \frac{1}{3}(N_2\left(\frac{h}{2}\right) - N_2(h))$.
   (a) $N(0.04) = 2.665836331$, $N(0.02) = 2.691588029$, $N(0.01) = 2.704813829$
   (b) $N_2(0.04) = 2.717339727$, $N_2(0.02) = 2.718039629$. The $O(h^3)$ approximation is $N_3(0.04) = 2.718272931$.
   (c) Yes, since the errors seem proportioned to h for $N(h)$, to h^2 for $N_2(h)$, and to h^3 for $N_3(h)$.
12. (a) We have

$$\lim_{h\to 0} \frac{\ln(2+h) - \ln(2-h)}{h} = \lim_{h\to 0} \frac{1}{2+h} + \frac{1}{2-h} = 1,$$

so

$$\lim_{h\to 0}\left(\frac{2+h}{2-h}\right)^{1/h} = \lim_{h\to 0} e^{\frac{1}{h}[\ln(2+h)-\ln(2-h)]} = e^1 = e.$$

(b) $N(0.04) = 2.718644377221219$, $N(0.02) = 2.718372444800607$, $N(0.01) = 2.718304481241685$

(c) Let $N_2(h) = 2N\left(\frac{h}{2}\right) - N(h)$ and $N_3(h) = N_2\left(\frac{h}{2}\right) + \frac{1}{3}\left[N_2\left(\frac{h}{2}\right) - N_2(h)\right]$. Then $N_2(0.04) = 2.718100512379995$, $N_2(0.02) = 2.718236517682763$ and $N_3(0.04) = 2.718281852783685$. $N_3(0.04)$ is an $O\left(h^3\right)$ approximation satisfying $|e - N_3(0.04)| \leq 0.5 \times 10^{-7}$.

(d)

$$N(-h) = \left(\frac{2-h}{2+h}\right)^{1/-h} = \left(\frac{2+h}{2-h}\right)^{1/h} = N(h)$$

(e) Let

$$e = N(h) + K_1 h + K_2 h^2 + K_3 h^3 + \cdots .$$

Replacing h by $-h$ gives

$$e = N(-h) - K_1 h + K_2 h^2 - K_3 h^3 + \cdots ,$$

but $N(-h) = N(h)$, so that

$$e = N(h) - K_1 h + K_2 h^2 - K_3 h^3 + \cdots .$$

Thus,

$$K_1 h + K_3 h^3 + \cdots = -K_1 h - K_3 h^3 \cdots ,$$

and it follows that $K_1 = K_3 = K_5 = \cdots = 0$ and

$$e = N(h) + K_2 h^2 + K_4 h^4 + \cdots .$$

(f) Let

$$N_2(h) = N\left(\frac{h}{2}\right) + \frac{1}{3}\left(N\left(\frac{h}{2}\right) - N(h)\right)$$

and

$$N_3(h) = N_2\left(\frac{h}{2}\right) + \frac{1}{15}\left(N_2\left(\frac{h}{2}\right) - N_2(h)\right).$$

Then

$$N_2(0.04) = 2.718281800660402, N_2(0.02) = 2.718281826722043$$

and

$$N_3(0.04) = 2.718281828459487.$$

$N_3(0.04)$ is an $O\left(h^6\right)$ approximation satisfying

$$|e - N_3(0.04)| \leq 0.5 \times 10^{-12}.$$

13. (a) We have

$$P_{0,1}(x) = \frac{\left(x-h^2\right) N_1\left(\frac{h}{2}\right)}{\frac{h^2}{4} - h^2} + \frac{\left(x - \frac{h^2}{4}\right) N_1(h)}{h^2 - \frac{h^2}{4}}, \quad \text{so} \quad P_{0,1}(0) = \frac{4N_1\left(\frac{h}{2}\right) - N_1(h)}{3}.$$

Similarly,

$$P_{1,2}(0) = \frac{4N_1\left(\frac{h}{4}\right) - N_1\left(\frac{h}{2}\right)}{3}.$$

(b) We have

$$P_{0,2}(x) = \frac{(x - h^4)\, N_2\left(\frac{h}{2}\right)}{\frac{h^4}{16} - h^4} + \frac{\left(x - \frac{h^4}{16}\right) N_2(h)}{h^4 - \frac{h^4}{16}}, \quad \text{so} \quad P_{0,2}(0) = \frac{16N_2\left(\frac{h}{2}\right) - N_2(h)}{15}.$$

14. All the approximations of the form $N_{2i}\left(h/2^j\right)$, for $i = 1, 2, \ldots$ and $j = 0, 1, 2, \ldots$, will be upper bounds for M, and all the approximations of the form $N_{2i+1}\left(\frac{h}{2^j}\right)$, for $i = 0, 1, 2, \ldots$ and $j = 0, 1, 2, \ldots$, will be lower bounds for M.

15. (a) The polygonal approximations are in the following table.

k	4	8	16	32	64	128	256	512
p_k	$2\sqrt{2}$	3.0614675	3.1214452	3.1365485	3.1403312	3.1412723	3.1415138	3.1415729
P_k	4	3.3137085	3.1825979	3.1517249	3.144184	3.1422236	3.1417504	3.1416321

(b) Values of p_k and P_k are given in the following tables, together with the extrapolation results:

For p_k we have :

2.8284271				
3.0614675	3.1391476			
3.1214452	3.1414377	3.1415904		
3.1365485	3.1415829	3.1415926	3.1415927	
3.1403312	3.1415921	3.1415927	3.1415927	3.1415927

For P_k we have :

4				
3.3137085	3.0849447			
3.1825979	3.1388943	3.1424910		
3.1517249	3.1414339	3.1416032	3.1415891	
3.1441184	3.1415829	3.1415928	3.1415926	3.1415927

Exercise Set 4.3, page 195

1. The Trapezoidal rule gives the following approximations.

(a) 0.265625 (b) −0.2678571 (c) −0.17776434 (d) 0.1839397

(e) −0.8666667 (f) −0.1777643 (g) 0.2180895 (h) 4.1432597