Hw 10

1\. To assign one cubic polynomial on each of the intervals $[x_0, x_1], [x_1, x_2], \dots, [x_{n-2}, x_{n-1}], [x_{n-1}, x_n]$, with the usual continuity conditions across the "knots" $x_1, x_2, \dots, x_{n-1}$ and two extra conditions

$$S_1'''(x_1) = S_2'''(x_1) \quad \text{and} \quad S_{n-1}'''(x_{n-1}) = S_{n-2}'''(x_{n-1})$$

where $S(x) = S_i(x)$ when $x_{i-1} \le x \le x_i$, $i = 1, 2, \dots, n$

Consider $S_1(x)$ and $S_2(x)$

Since $S_1^{(k)}(x_1) = S_2^{(k)}(x_1)$, $k = 0, 1, 2, 3$ and $\deg(S_1) \le 3$, $\deg(S_2) \le 3$

$\Rightarrow S_1(x) = S_2(x)$, $\forall x \in \mathbb{R}$

Similarly, $S_{n-1}(x) = S_n(x)$, $\forall x \in \mathbb{R}$

Hence, we can rewrite
$$S(x) = \begin{cases} S_1(x), & \text{when } x_0 \le x \le x_2 \\ S_i(x), & \text{when } x_{i-1} \le x \le x_i, \quad i = 3, 4, \dots, n-2 \\ S_{n-1}(x), & \text{when } x_{n-2} \le x \le x_n \end{cases}$$

and S has to satisfy the following condition

(i) $S_1(x_0) = f(x_0)$, $S_1(x_1) = f(x_1)$, $S_1(x_2) = f(x_2)$

$S_i(x_{i-1}) = f(x_{i-1})$, $S_i(x_i) = f(x_i)$, $i = 3, 4, \dots, n-2$

$S_{n-1}(x_{n-2}) = f(x_{n-2})$, $S_{n-1}(x_{n-1}) = f(x_{n-1})$, $S_{n-1}(x_n) = f(x_n)$

(ii) $S_1'(x_2) = S_3'(x_2)$, $S_i'(x_i) = S_{i+1}'(x_i)$, $i = 3, 4, \dots, n-3$, $S_{n-2}'(x_{n-2}) = S_{n-1}'(x_{n-2})$

(iii) $S_1''(x_2) = S_3''(x_2)$, $S_i''(x_i) = S_{i+1}''(x_i)$, $i = 3, 4, \cdots, n-3$, $S_{n-2}''(x_{n-2}) = S_{n-1}''(x_{n-2})$

Which is equivalent to assign one cubic polynomial on each of the intervals $[x_0, x_2]$, $[x_2, x_3]$, $\cdots$, $[x_{n-3}, x_{n-2}]$, $[x_{n-2}, x_n]$ with the usual continuity conditions across the "knots" $x_2, x_3, \cdots, x_{n-2}$ and two extra conditions

$$S(x_1) = f(x_1), \quad S(x_{n-1}) = f(x_{n-1})$$