19. Since $f[x_2] = f[x_0] + f[x_0, x_1](x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$,

$$a_2 = \frac{f[x_2] - f[x_0]}{(x_2 - x_0)(x_2 - x_1)} - \frac{f[x_0, x_1]}{(x_2 - x_1)}.$$

This simplifies to $f[x_0, x_1, x_2]$.

20. Theorem 3.3 gives

$$f(x) = P_n(x) + \frac{f^{n+1}(\xi(x))}{(n+1)!}(x - x_0)\dots(x - x_n).$$

Let $x_{n+1} = x$. The interpolation polynomial of degree $n+1$ on $x_0, x_1, \dots, x_{n+1}$ is

$$P_{n+1}(t) = P_n(t) + f[x_0, x_1, ..., x_n, x_{n+1}](t - x_0)(t - x_1)\dots(t - x_n).$$

Since $f(x) = P_{n+1}(x)$, we have

$$P_n(x) + \frac{f^{n+1}(\xi(x))}{(n+1)!}(x - x_0)\dots(x - x_n) = P_n(x) + f[x_0, ..., x_n, x](x - x_0)\dots(x - x_n).$$

Thus,

$$f[x_0, ..., x_n, x] = \frac{f^{n+1}(\xi(x))}{(n+1)!}.$$

21. Let $\tilde{P}(x) = f[x_{i_0}] + \sum_{k=1}^{n} f[x_{i_0}, \dots, x_{i_k}](x - x_{i_0})\cdots(x - x_{i_k})$ and $\hat{P}(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, \dots, x_k](x - x_0)\cdots(x - x_k)$. The polynomial $\tilde{P}(x)$ interpolates $f(x)$ at the nodes $x_{i_0}, \dots, x_{i_n}$, and the polynomial $\hat{P}(x)$ interpolates $f(x)$ at the nodes $x_0, \dots, x_n$. Since both sets of nodes are the same and the interpolating polynomial is unique, we have $\tilde{P}(x) = \hat{P}(x)$. The coefficient of x^n in $\tilde{P}(x)$ is $f[x_{i_0}, \dots, x_{i_n}]$, and the coefficient of x^n in $\hat{P}(x)$ is $f[x_0, \dots, x_n]$. Thus, $f[x_{i_0}, \dots, x_{i_n}] = f[x_0, \dots, x_n]$.

Exercise Set 3.3, page 135

1. The coefficients for the polynomials in divided-difference form are given in the following tables. For example, the polynomial in part (a) is

$$H_3(x) = 17.56492 + 3.116256(x - 8.3) + 0.05948(x - 8.3)^2 - 0.00202222(x - 8.3)^2(x - 8.6).$$

(a)	(b)	(c)	(d)
17.56492	0.22363362	−0.02475	−0.62049958
3.116256	2.1691753	0.751	3.5850208
0.05948	0.01558225	2.751	−2.1989182
−0.00202222	−3.2177925	1	−0.490447
		0	0.037205
		0	0.040475
			−0.0025277777
			0.0029629628

2. The coefficients for the polynomials in divided-difference form are given in the following tables. For example, the polynomial in part (a) is $H_3(x) = 1 + 2x + 2.87312x^2 + 2.25376x^2(x - 0.5)$

(a)	(b)	(c)	(d)
1.0	1.33203	−0.29004996	0.8619948
2.0	0.4375	−2.8019975	0.1553624
2.87312	−2.999996	0.945237	0.07337636
2.25376	7.749984	−0.297	0.01583112
		−0.47935	−0.00014728
		0.05	−0.00089244
			−0.00007672
			0.00005975111111

3. The following table shows the approximations.

	x	Approximation to $f(x)$	Actual $f(x)$	Error
(a)	8.4	17.877144	17.877146	2.33×10^{-6}
(b)	0.9	0.44392477	0.44359244	3.3323×10^{-4}
(c)	$-\frac{1}{3}$	0.1745185	0.17451852	1.85×10^{-8}
(d)	0.25	−0.1327719	−0.13277189	5.42×10^{-9}

4. The following table shows the approximations.

	x	Approximation to $f(x)$	Actual $f(x)$	Error
(a)	0.43	2.362069472	2.363160694	0.001091222
(b)	0.0	1.132811175	1.000000000	0.132811750
(c)	0.18	-0.5081234697	-0.5081234644	0.53×10^{-8}
(d)	0.25	1.189069883	1.189069931	0.48×10^{-7}

5. (a) We have $\sin 0.34 \approx H_5(0.34) = 0.33349$.
   (b) The formula gives an error bound of 3.05×10^{-14}, but the actual error is 2.91×10^{-6}. The discrepancy is due to the fact that the data are given to only five decimal places.

(c) We have $\sin 0.34 \approx H_7(0.34) = 0.33350$. Although the error bound is now 5.4×10^{-20}, the accuracy of the given data dominates the calculations. This result is actually less accurate than the approximation in part (b), since $\sin 0.34 = 0.333487$.

6. (a) $H(1.03) = 0.80932485$. The actual error is 1.24×10^{-6}, and error bound is 1.31×10^{-6}.

   (b) $H(1.03) = 0.809323619263$. The actual error is 3.63×10^{-10}, and an error bound is 3.86×10^{-10}.

7. For 3(a), we have an error bound of 5.9×10^{-8}. The error bound for 3(c) is 0 since $f^{(n)}(x) \equiv 0$, for $n > 3$.

8. For 4(a), we have an error bound of 1.6×10^{-3}. The error bound for 4(c) is 1.5×10^{-7}.

9. $H_3(1.25) = 1.169080403$ with an error bound of 4.81×10^{-5}, and $H_5(1.25) = 1.169016064$ with an error bound of 4.43×10^{-4}.

10. The Hermite polynomial generated from these data is

$$\begin{aligned} H_9(x) =& 75x + 0.222222x^2(x-3) - 0.0311111x^2(x-3)^2 \\ & - 0.00644444x^2(x-3)^2(x-5) + 0.00226389x^2(x-3)^2(x-5)^2 \\ & - 0.000913194x^2(x-3)^2(x-5)^2(x-8) + 0.000130527x^2(x-3)^2(x-5)^2(x-8)^2 \\ & - 0.0000202236x^2(x-3)^2(x-5)^2(x-8)^2(x-13). \end{aligned}$$

   (a) The Hermite polynomial predicts a position of $H_9(10) = 743$ ft and a speed of $H_9'(10) = 48$ ft/sec. Although the position approximation is reasonable, the low speed prediction is suspect.

   (b) To find the first time the speed exceeds 55 mi/hr, which is equivalent to $80.\bar{6}$ ft/sec, we solve for the smallest value of t in the equation $80.\bar{6} = H_9'(x)$. This gives $x \approx 5.6488092$.

   (c) The estimated maximum speed is $H_9'(12.37187) = 119.423$ ft/sec ≈ 81.425 mi/hr.

11. (a) Suppose $P(x)$ is another polynomial with $P(x_k) = f(x_k)$ and $P'(x_k) = f'(x_k)$, for $k = 0, \ldots, n$, and the degree of $P(x)$ is at most $2n+1$. Let

$$D(x) = H_{2n+1}(x) - P(x).$$

   Then $D(x)$ is a polynomial of degree at most $2n+1$ with $D(x_k) = 0$, and $D'(x_k) = 0$, for each $k = 0, 1, \ldots, n$. Thus, D has zeros of multiplicity 2 at each x_k and

$$D(x) = (x-x_0)^2 \ldots (x-x_n)^2 Q(x).$$

   Hence, $D(x)$ must be of degree $2n$ or more, which would be a contradiction, or $Q(x) \equiv 0$ which implies that $D(x) \equiv 0$. Thus, $P(x) \equiv H_{2n+1}(x)$.

   (b) First note that the error formula holds if $x = x_k$ for any choice of ξ.
   Let $x \neq x_k$, for $k = 0, \ldots, n$, and define

$$g(t) = f(t) - H_{2n+1}(t) - \frac{(t-x_0)^2 \ldots (t-x_n)^2}{(x-x_0)^2 \ldots (x-x_n)^2}\left[f(x) - H_{2n+1}(x)\right].$$

   Note that $g(x_k) = 0$, for $k = 0, \ldots, n$, and $g(x) = 0$. Thus, g has $n+2$ distinct zeros in $[a, b]$. By Rolle's Theorem, g' has $n+1$ distinct zeros $\xi_0, \ldots, \xi_n$, which are between the

numbers $x_0, ..., x_n, x$.
In addition, $g'(x_k) = 0$, for $k = 0, ..., n$, so g' has $2n+2$ distinct zeros $\xi_0, ..., \xi_n, x_0, ..., x_n$. Since g' is $2n+1$ times differentiable, the Generalized Rolle's Theorem implies that a number ξ in$[a, b]$ exists with $g^{(2n+2)}(\xi) = 0$. But,

$$g^{(2n+2)}(t) = f^{(2n+2)}(t) - \frac{d^{2n+2}}{dt^{2n+2}} H_{2n+1}(t) - \frac{[f(x) - H_{2n+1}(x)] \cdot (2n+2)!}{(x-x_0)^2 \cdots (x-x_n)^2}$$

and

$$0 = g^{(2n+2)}(\xi) = f^{(2n+2)}(\xi) - \frac{(2n+2)![f(x) - H_{2n+1}(x)]}{(x-x_0)^2 \cdots (x-x_n)^2}.$$

The error formula follows.

12. Let

$$H(x) = f[z_0] + f[z_0, z_1](x - x_0) + f[z_0, z_1, z_2](x-x_0)^2 + f[z_0, z_1, z_2, z_3](x-x_0)^2(x-x_1).$$

Substituting $f[z_0] = f(x_0)$, $f[z_0, z_1] = f'(x_0)$,

$$f[z_0, z_1, z_2] = \frac{f(x_1) - f(x_0) - f'(x_0)(x_1 - x_0)}{x_1 - x_0},$$

and

$$f[z_0, z_1, z_2, z_3] = \frac{f'(x_1)(x_1 - x_0) - 2f(x_1) + 2f(x_0) + f'(x_0)(x_1 - x_0)}{(x_1 - x_0)^3}$$

into $H(x)$ and simplifying gives

$$\begin{aligned} H(x) =& f(x_0) + f'(x_0)(x - x_0) + \frac{f(x_1) - f(x_0) - f'(x_0)(x_1 - x_0)}{(x_1 - x_0)^2}(x - x_0)^2 \\ &+ \frac{f'(x_1)(x_1 - x_0) - 2f(x_1) + 2f(x_0) + f'(x_0)(x_1 - x_0)}{(x_1 - x_0)^3}(x-x_0)^2(x-x_1). \end{aligned}$$

Thus, $H(x_0) = f(x_0)$ and

$$H(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + [f(x_1) - f(x_0) - f'(x_0)(x_1 - x_0)] = f(x_1).$$

Further,

$$\begin{aligned} H'(x) =& f'(x_0) + 2\frac{f(x_1) - f(x_0) - f'(x_0)(x_1 - x_0)}{(x_1 - x_0)^2}(x - x_0) \\ &+ \frac{f'(x_1)(x_1 - x_0) - 2f(x_1) + 2f(x_0) + f'(x_0)(x_1 - x_0)}{(x_1 - x_0)^3}[2(x-x_0)(x-x_1) + (x-x_0)^2], \end{aligned}$$

so

$$H'(x_0) = f'(x_0)$$

and

$$\begin{aligned} H'(x_1) =& f'(x_0) + \frac{2f(x_1)}{x_1 - x_0} - \frac{2f(x_0)}{x_1 - x_0} - 2f'(x_0) + f'(x_1) - \frac{2f(x_1)}{x_1 - x_0} + \frac{2f(x_0)}{x_1 - x_0} + f'(x_0) \\ =& f'(x_1). \end{aligned}$$

Thus, H satisfies the requirements of the cubic Hermite polynomial H_3, and the uniqueness of H_3 implies $H_3 = H$.

Exercise Set 3.4, page 153

1. We have $S(x) = x$ on $[0, 2]$.
2. We have $s(x) = x$ on $[0, 2]$.
3. The equations of the respective free cubic splines are

$$S(x) = S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the coefficients are given in the following tables.

(a)

i	a_i	b_i	c_i	d_i
0	17.564920	3.13410000	0.00000000	0.00000000

(b)

i	a_i	b_i	c_i	d_i
0	0.22363362	2.17229175	0.00000000	0.00000000

(c)

i	a_i	b_i	c_i	d_i
0	−0.02475000	1.03237500	0.00000000	6.50200000
1	0.33493750	2.25150000	4.87650000	−6.50200000

(d)

i	a_i	b_i	c_i	d_i
0	−0.62049958	3.45508693	0.00000000	−8.9957933
1	−0.28398668	3.18521313	−2.69873800	−0.94630333
2	0.00660095	2.61707643	−2.98262900	9.9420966

4. The equations of the respective free cubic splines are

$$s(x) = s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

for x in $[x_i, x_{i+1}]$, where the coefficients are given in the following table.

	i	a_i	b_i	c_i	d_i
(a)	0	1.00000000	3.43656000	0.00000000	0.00000000
(b)	0	1.33203000	-1.06249800	0.00000000	0.00000000
(c)	0	-0.29004996	-2.75128630	0.00000000	4.38125000
	1	-0.56079734	-2.61984880	1.31437500	-4.38125000
(d)	0	0.86199480	0.17563785	0.00000000	0.06565093
	1	0.95802009	0.22487604	0.09847639	0.02828072
	2	1.09861230	0.34456298	0.14089747	−0.09393165

5. The following tables show the approximations.

	x	Approximation to $f(x)$	Actual $f(x)$	Error
(a)	8.4	17.87833	17.877146	1.1840×10^{-3}
(b)	0.9	0.4408628	0.44359244	2.7296×10^{-3}
(c)	$-\frac{1}{3}$	0.1774144	0.17451852	2.8959×10^{-3}
(d)	0.25	−0.1315912	−0.13277189	1.1807×10^{-3}

	x	Approximation to $f'(x)$	Actual $f'(x)$	Error
(a)	8.4	3.134100	3.128232	5.86829×10^{-3}
(b)	0.9	2.172292	2.204367	0.0320747
(c)	$-\frac{1}{3}$	1.574208	1.668000	0.093792
(d)	0.25	2.908242	2.907061	1.18057×10^{-3}

6. The following tables show the approximations.

	x	$f(x)$	$s(x)$	Error
(a)	0.43	2.363160694	2.4777208	0.114560106
(b)	0.0	1.000000000	1.066405500	0.066405500
(c)	0.18	−0.5081234644	−0.5079096640	0.0002138004
(d)	0.25	1.189069931	1.192091455	0.003021524

	x	$f'(x)$	$s'(x)$	Error
(a)	0.43	4.726321388	3.436560000	1.289761388
(b)	0.0	−1.000000000	−1.06249800	0.06249800
(c)	0.18	−2.651616829	−2.66716630	0.015549471
(d)	0.25	0.3909913152	0.3973995306	0.0064082154

7. The equations of the respective clamped cubic splines are

$$s(x) = s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the coefficients are given in the following tables.

(a)

i	a_i	b_i	c_i	d_i
0	17.564920	3.1162560	0.0600867	−0.00202222

(b)

i	a_i	b_i	c_i	d_i
0	0.22363362	2.1691753	0.65914075	−3.2177925

(c)

i	a_i	b_i	c_i	d_i
0	−0.02475000	0.75100000	2.5010000	1.0000000
1	0.33493750	2.18900000	3.2510000	1.0000000

(d)

i	a_i	b_i	c_i	d_i
0	−0.62049958	3.5850208	−2.1498407	−0.49077413
1	−0.28398668	3.1403294	−2.2970730	−0.47458360
2	0.006600950	2.6666773	−2.4394481	−0.44980146

8. The coefficients of the clamped cubic spline interpolation are given in the following table.

	i	a_i	b_i	c_i	d_i
(a)	0	1.00000000	2.00000000	1.74624000	2.25376000
(b)	0	1.33203000	0.43750000	−6.87498800	7.74998400
(c)	0	−0.29004996	−2.80199750	0.97498700	−0.29750000
	1	−0.56079734	−2.61592510	0.88573700	−0.48724000
(d)	0	0.86199480	0.15536240	0.06537475	0.01600323
	1	0.95802009	0.23273957	0.08937959	0.01502024
	2	1.09861230	0.33338433	0.11190995	0.00875797

9. $B = \frac{1}{4}$, $D = \frac{1}{4}$, $b = -\frac{1}{2}$, $d = \frac{1}{4}$

10. The following tables show the approximations.

	x	$f(x)$	$s(x)$	Error
(a)	0.43	2.363160694	2.362069472	0.001091222
(b)	0.0	1.000000000	1.132811750	0.132811750
(c)	0.18	−0.5081234644	−0.4443014992	0.0638219652
(d)	0.25	1.189069931	1.189089597	0.000019666

	x	$f'(x)$	$s'(x)$	Error
(a)	0.43	4.726321388	4.751927072	0.025605684
(b)	0.0	−1.000000000	−1.546872000	0.546872000
(c)	0.18	−2.651616829	−2.325976780	0.325640049
(d)	0.25	0.3909913152	0.3909814244	0.98908×10^{-5}

11. $b = -1$, $c = -3$, $d = 1$

12. $a = 4$, $b = 4$, $c = -1$, $d = \frac{1}{3}$

13. $B = \frac{1}{4}$, $D = \frac{1}{4}$, $b = -\frac{1}{2}$, $d = \frac{1}{4}$

14. $f'(0) = 0,\ f'(2) = 11$

15. (a) The equation of the spline is

$$S(x) = S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the coefficients are given in the following table.

x_i	a_i	b_i	c_i	d_i
0	1.0	−0.7573593	0.0	−6.627417
0.25	0.7071068	−2.0	−4.970563	6.627417
0.5	0.0	−3.242641	0.0	6.627417
0.75	−0.7071068	−2.0	4.970563	−6.627417

(b) $\int_0^1 S(x)dx = 0.000000$ (c) $S'(0.5) = -3.24264$ (d) $S''(0.5) = 0.0$

16. The equation of the spline is

$$S(x) = S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the results are given in the following table.

x_i	a_i	b_i	c_i	d_i
0	1.00000	−0.923601	0	0.620865
0.25	0.778801	−0.807189	0.465649	−0.154017
0.75	0.472367	−0.457052	0.234624	−0.312832

We have $\int_0^1 S(x)\ dx = 0.631967$, $S'(0.5) = -0.603243$, and $S''(0.5) = 0.700274$. Also, $\int_0^1 e^{-x}\ dx = 0.63212056$, $f'(0.5) = -0.6065307$, and $f''(0.5) = 0.6065307$.

17. The equation of the spline is

$$s(x) = s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the coefficients are given in the following table.

x_i	a_i	b_i	c_i	d_i
0	1.0	0.0	−5.193321	2.028118
0.25	0.7071068	−2.216388	−3.672233	4.896310
0.5	0.0	−3.134447	0.0	4.896310
0.75	−0.7071068	−2.216388	3.672233	2.028118

$\int_0^1 s(x)\,dx = 0.000000$, $s'(0.5) = -3.13445$, and $s''(0.5) = 0.0$

18. The equation of the spline is

$$s(x) = s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the coefficients are given in the following table.

x_i	a_i	b_i	c_i	d_i
0	1.00000	−1.00000	0.499440	−0.154515
0.25	0.778801	−0.779251	0.383555	−0.101580
0.75	0.472367	−0.471881	0.231185	−0.0618174

We have $\int_0^1 s(x)\,dx = 0.623078$, $s'(0.5) = -0.606520$, and $s''(0.5) = 0.614740$. Also, $\int_0^1 e^{-x}\,dx = 0.6321205$, $f'(0.5) = -0.6065307$, and $f''(0.5) = 0.6065307$.

19. Let $f(x) = a + bx + cx^2 + dx^3$. Clearly, f satisfies properties (a), (c), (d), and (e) of Definition 3.10, and f interpolates itself for any choice of $x_0, \ldots, x_n$. Since (ii) of property (f) in Definition 3.10 holds, f must be its own clamped cubic spline. However, $f''(x) = 2c + 6dx$ can be zero only at $x = -c/3d$. Thus, part (i) of property (f) in Definition 3.10 cannot hold at two values x_0 and x_n. Thus, f cannot be a natural cubic spline.

20. The free cubic spline must be the linear function $L(x)$ through all the data $\{x_i, f(x_i)\}_{i=1}^n$ since $L''(x) = 0$ for all x. So properties (a), (b), (c), (d), (e), (f), (i) of Definition 3.10 would be satisfied.
If f is linear, then f is its own clamped cubic spline. If, for example, f satisfies $f(0) = 0$, $f(1) = 1$, $f(2) = 2$, $f'(0) = 1$, and $f'(2) = 0$, then the data lie on a straight line but the function f is not linear. In that case the spline is

$$s(x) = \begin{cases} x - \frac{1}{4}x^2 + \frac{1}{4}x^3, & 0 \le x \le 1 \\ 1 + \frac{5}{4}(x-1) + \frac{1}{2}(x-1)^2 - \frac{3}{4}(x-1)^3, & 1 \le x \le 2, \end{cases}$$

which is not a linear function.

21. The piecewise linear approximation to f is given by

$$F(x) = \begin{cases} 20(e^{0.1} - 1)x + 1, & \text{for } x \text{ in } [0, 0.05] \\ 20(e^{0.2} - e^{0.1})x + 2e^{0.1} - e^{0.2}, & \text{for } x \text{ in } (0.05, 1]. \end{cases}$$

We have

$$\int_0^{0.1} F(x)\,dx = 0.1107936 \quad \text{and} \quad \int_0^{0.1} f(x)\,dx = 0.1107014.$$

22. $|f(x) - F(x)| \le \frac{M}{8} \max_{0 \le j \le n-1} |x_{j+1} - x_j|^2$, where $M = \max_{a \le x \le b} |f''(x)|$.
Error bounds for Exercise 21 are on $[0, 0.1]$, $|f(x) - F(x)| \le 1.53 \times 10^{-3}$ and

$$\left| \int_0^{0.1} F(x)\,dx - \int_0^{0.1} e^{2x}\,dx \right| \le 1.53 \times 10^{-4}.$$

23. Insert the following before Step 7 in Algorithm 3.4 and Step 8 in Algorithm 3.5:

For $j = 0, 1, \ldots, n-1$ set

$$l_1 = b_j; \text{ (Note that } l_1 = s'(x_j).)$$
$$l_2 = 2c_j; \text{ (Note that } l_2 = s''(x_j).)$$
$$\text{OUTPUT } (l_1, l_2)$$

Set

$$l_1 = b_{n-1} + 2c_{n-1}h_{n-1} + 3d_{n-1}h_{n-1}^2; \text{(Note that } l_1 = s'(x_n).)$$
$$l_2 = 2c_{n-1} + 6d_{n-1}h_{n-1}; \text{(Note that } l_2 = s''(x_n).)$$
$$\text{OUTPUT } (l_1, l_2).$$

24. Before STEP 7 in Algorithm 3.4 and STEP 8 in Algorithm 3.5 insert the following:

Set $I = 0$;

For $j = 0, \ldots, n-1$ set

$$I = a_j h_j + \tfrac{b_j}{2} h_j^2 + \tfrac{c_j}{3} h_j^3 + \tfrac{d_j}{4} h_j^4 + I. \left(\textit{Accumulate } \int_{x_j}^{x_{j+1}} S(x)\,dx.\right)$$

OUTPUT (I).

25. (a) On $[0, 0.05]$, we have $s(x) = 1.000000 + 1.999999x + 1.998302x^2 + 1.401310x^3$, and on $(0.05, 0.1]$, we have $s(x) = 1.105170 + 2.210340(x - 0.05) + 2.208498(x - 0.05)^2 + 1.548758(x - 0.05)^3$.

(b) $\int_0^{0.1} s(x)\,dx = 0.110701$

(c) 1.6×10^{-7}

(d) On $[0, 0.05]$, we have $S(x) = 1 + 2.04811x + 22.12184x^3$, and on $(0.05, 0.1]$, we have $S(x) = 1.105171 + 2.214028(x-0.05) + 3.318277(x-0.05)^2 - 22.12184(x-0.05)^3$. $S(0.02) = 1.041139$ and $S(0.02) = 1.040811$.

26. The five equations are $a_0 = f(x_0)$, $a_1 = f(x_1)$, $a_1 + b_1(x_2 - x_1) + c_1(x_2 - x_1)^2 = f(x_2)$, $a_0 + b_0(x_1 - x_0) + c_0(x_1 - x_0)^2 = a_1$, and $b_0 + 2c_0(x_1 - x_0) = b_1$.

If $S \in C^2$, then S is a quadratic on $[x_0, x_2]$ and the solution may not be meaningful.

27. We have

$$S(x) = \begin{cases} 2x - x^2, & 0 \le x \le 1 \\ 1 + (x-1)^2, & 1 \le x \le 2 \end{cases}$$

28. (a)

x_i	a_i	b_i	c_i	d_i
1940	132165	1651.85	0.00000	2.64248
1950	151326	2444.59	79.2744	−4.37641
1960	179323	2717.16	−52.0179	2.00918
1970	203302	2279.55	8.25746	−0.381311
1980	226542	2330.31	−3.18186	0.106062

$S(1930) = 113004$, $S(1965) = 191860$, and $S(2010) = 296451$.

(b) Probably not very accurate.

29. The spline has the equation

$$s(x) = s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the coefficients are given in the following table.

x_i	a_i	b_i	c_i	d_i
0	0	75	−0.659292	0.219764
3	225	76.9779	1.31858	−0.153761
5	383	80.4071	0.396018	−0.177237
8	623	77.9978	−1.19912	0.0799115

The spline predicts a position of $s(10) = 774.84$ ft and a speed of $s'(10) = 74.16$ ft/s. To maximize the speed, we find the single critical point of $s'(x)$, and compare the values of $s(x)$ at this point and the endpoints. We find that max $s'(x) = s'(5.7448) = 80.7$ ft/s $= 55.02$ mi/h. The speed 55 mi/h was first exceeded at approximately 5.5 s.

30. (a) The coefficients are given in the following table.

a_i	b_i	c_i	d_i
0.00000000	91.39016393	0.00000000	9.11737705
22.99000000	93.09967213	6.83803279	2.41311476
46.73000000	96.97114754	8.64786886	-.22032787
97.35000000	105.45377050	8.31737705	-11.08983607

(b) The predicted time at the three-quarter mile pole was 1 : 24.48.

(c) The starting speed is predicted to be 39.39 mi/h and the speed at the finish line is predicted to be 33.48 mi/h.

31. The equation of the spline is

$$S(x) = S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the coefficients are given in the following table.

	Sample 1				Sample 2			
x_i	a_i	b_i	c_i	d_i	a_i	b_i	c_i	d_i
0	6.67	−0.44687	0	0.06176	6.67	1.6629	0	−0.00249
6	17.33	6.2237	1.1118	−0.27099	16.11	1.3943	−0.04477	−0.03251
10	42.67	2.1104	−2.1401	0.28109	18.89	−0.52442	−0.43490	0.05916
13	37.33	−3.1406	0.38974	−0.01411	15.00	−1.5365	0.09756	0.00226
17	30.10	−0.70021	0.22036	−0.02491	10.56	−0.64732	0.12473	−0.01113
20	29.31	−0.05069	−0.00386	0.00016	9.44	−0.19955	0.02453	−0.00102

32. The three clamped splines have equations of the form

$$s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the values of the coefficients are given in the following tables.

Spline 1

i	x_i	$a_i = f(x_i)$	b_i	c_i	d_i	$f'(x_i)$
0	1	3.0	1.0	−0.347	−0.049	1.0
1	2	3.7	0.447	−0.206	0.027	
2	5	3.9	−0.074	0.033	0.342	
3	6	4.2	1.016	1.058	−0.575	
4	7	5.7	1.409	−0.665	0.156	
5	8	6.6	0.547	−0.196	0.024	
6	10	7.1	0.048	−0.053	−0.003	
7	13	6.7	−0.339	−0.076	0.006	
8	17	4.5				−0.67

Spline 2

i	x_i	$a_i = f(x_i)$	b_i	c_i	d_i	$f'(x_i)$
0	17	4.5	3.0	−1.101	−0.126	3.0
1	20	7.0	−0.198	0.035	−0.023	
2	23	6.1	−0.609	−0.172	0.280	
3	24	5.6	−0.111	0.669	−0.357	
4	25	5.8	0.154	−0.403	0.088	
5	27	5.2	−0.401	0.126	−2.568	
6	27.7	4.1				−4.0

Spline 3

i	x_i	$a_i = f(x_i)$	b_i	c_i	d_i	$f'(x_i)$
0	27.7	4.1	0.330	2.262	−3.800	0.33
1	28	4.3	0.661	−1.157	0.296	
2	29	4.1	−0.765	−0.269	−0.065	
3	30	3.0				−1.5

33. The three natural splines have equations of the form

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the values of the coefficients are given in the following tables.

Spline 1

i	x_i	$a_i = f(x_i)$	b_i	c_i	d_i
0	1	3.0	0.786	0.0	−0.086
1	2	3.7	0.529	−0.257	0.034
2	5	3.9	−0.086	0.052	0.334
3	6	4.2	1.019	1.053	−0.572
4	7	5.7	1.408	−0.664	0.156
5	8	6.6	0.547	−0.197	0.024
6	10	7.1	0.049	−0.052	−0.003
7	13	6.7	−0.342	−0.078	0.007
8	17	4.5			

Spline 2

i	x_i	$a_i = f(x_i)$	b_i	c_i	d_i
0	17	4.5	1.106	0.0	−0.030
1	20	7.0	0.289	−0.272	0.025
2	23	6.1	−0.660	−0.044	0.204
3	24	5.6	−0.137	0.567	−0.230
4	25	5.8	0.306	−0.124	−0.089
5	27	5.2	−1.263	−0.660	0.314
6	27.7	4.1			

Spline 3

i	x_i	$a_i = f(x_i)$	b_i	c_i	d_i
0	27.7	4.1	0.749	0.0	−0.910
1	28	4.3	0.503	−0.819	0.116
2	29	4.1	−0.787	−0.470	0.157
3	30	3.0			

Exercise Set 3.5, page 163

1. The parametric cubic Hermite approximations are as follows.

   (a) $x(t) = -10t^3 + 14t^2 + t, \quad y(t) = -2t^3 + 3t^2 + t$

   (b) $x(t) = -10t^3 + 14.5t^2 + 0.5t, \quad y(t) = -3t^3 + 4.5t^2 + 0.5t$

   (c) $x(t) = -10t^3 + 14t^2 + t, \quad y(t) = -4t^3 + 5t^2 + t$