Hw 08

2. (i) $P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x)$ (3.1)

For each $0 \le i \le n$, we have one multiplication between $f(x_i)$ and $L_{n,i}(x)$

For $L_{n,k}(x) = \dfrac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_{n-1})(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_{n-1})(x_k-x_n)}$ $\leftarrow *$ (3.2)

There are $(n-1)\times 2+1 = 2n-1$ multiplications in $L_{n,k}(x)$, $0 \le k \le n$

Hence, for interpolation through evaluating (3.1) and (3.2),

we have $(2n-1)\times(n+1)+(n+1) = 2n(n+1)$ multiplications

(ii) In Algorithm 3.2, we have i multiplications for $F_{i,i}$, $1 \le i \le n$

$\Rightarrow 1+2+3+\cdots+n = \frac{n(n+1)}{2}$ multiplications

And $P_n(x) = \left(\cdots\left(\left(a_n(x-x_{n-1})+a_{n-1}\right)(x-x_{n-2})+a_{n-2}\right)\cdots\right)(x-x_0)+a_0$ (3.5)

In (3.5), we have n multiplications

Total: $\frac{n(n+1)}{2} + n = \frac{n(n+3)}{2}$ multiplications

3. Let $f(x) = L_{3,0}(x) + L_{3,1}(x) + L_{3,2}(x) + L_{3,3}(x) - 1$

Since $\deg(f) = 3$ and $f(x_0) = f(x_1) = f(x_2) = f(x_3) = 0$

$\Rightarrow f(x) = 0$

$\Rightarrow L_{3,0}(x) + L_{3,1}(x) + L_{3,2}(x) + L_{3,3}(x) = 1$, for all $x \in \mathbb{R}$

4. $(x_3, f(x_3)) = (0.5, 4) \Rightarrow f[x_2, x_3] = 10 \Rightarrow f[x_1, x_2, x_3] = 0 \Rightarrow f[x_0, x_1, x_2, x_3] = \frac{-100}{7}$

6. (a) Let $x_0, \cdots, x_n$ be uniformly spaced nodes on $[a,b]$, $x_j = a + jh$, $h = \frac{b-a}{n}$

$\forall\ x \in [a,b]$, $x_{j-1} \le x \le x_j$ for some $1 \le j \le n$

$$\Rightarrow |(x-x_{j-1})(x-x_j)| \le h^2 \quad \text{and} \quad \begin{cases} |x - x_{j-1-k}| \le (k+1)h, & 1 \le k \le j-1 \\ |x - x_{j+t}| \le (t+1)h \le (t+j)h, & 1 \le t \le n-j \end{cases}$$

$$\Rightarrow |(x-x_0) \cdots (x-x_{j-1})(x-x_j) \cdots (x-x_n)| \le |jh \cdot (j-1)h \cdot \cdots \cdot 2h \cdot h^2 \cdot (j+1)h \cdot (j+2)h \cdot \cdots \cdot nh|$$
$$= n!\, h^{n+1}, \quad \forall\ x \in [a,b]$$

(b)

$$|e^x - P_n(x)| = \left| \frac{e^{\xi_n}}{(n+1)!} (x-x_0)(x-x_1)\cdots(x-x_n) \right|, \text{ where } \xi_n \in (a,b)$$

$$\le \left| \frac{e^{\xi_n}}{n+1} \right| \cdot h^{n+1} \quad (\text{By (a)})$$

$$\Rightarrow \lim_{n\to\infty} \max_{0 \le x \le 1} |e^x - P_n(x)| \le \lim_{n\to\infty} \frac{e}{n+1} \cdot h^{n+1} = 0$$