Interpolation and Polynomial Approximation

Exercise Set 3.1, page 115

1. The interpolation polynomials are as follows.

   (a) $P_1(x) = -0.148878x + 1$; $P_1(0.45) = 0.933005$;
   $|f(0.45) - P_1(0.45)| = 0.032558$;
   $P_2(x) = -0.452592x^2 - 0.0131009x + 1$; $P_2(0.45) = 0.902455$;
   $|f(0.45) - P_2(0.45)| = 0.002008$

   (b) $P_1(x) = 0.467251x + 1$; $P_1(0.45) = 1.210263$;
   $|f(0.45) - P_1(0.45)| = 0.006104$;
   $P_2(x) = -0.0780026x^2 + 0.490652x + 1$; $P_2(0.45) = 1.204998$;
   $|f(0.45) - P_2(0.45)| = 0.000839$

   (c) $P_1(x) = 0.874548x$; $P_1(0.45) = 0.393546$;
   $|f(0.45) - P_1(0.45)| = 0.0212983$;
   $P_2(x) = -0.268961x^2 + 0.955236x$; $P_2(0.45) = 0.375392$;
   $|f(0.45) - P_2(0.45)| = 0.003828$

   (d) $P_1(x) = 1.031121x$; $P_1(0.45) = 0.464004$;
   $|f(0.45) - P_1(0.45)| = 0.019051$;
   $P_2(x) = 0.615092x^2 + 0.846593x$; $P_2(0.45) = 0.505523$;
   $|f(0.45) - P_2(0.45)| = 0.022468$

2. The interpolation polynomials are as follows.

   (a) $P_1(x) = -0.6969992408x + 0.1641422691$; $P_1(1.4) = -0.8116566680$;
   $|f(1.4) - P_1(1.4)| = 0.1393998486$;
   $P_2(x) = 3.552379809x^2 - 10.82128170x + 7.268901887$; $P_2(1.4) = -0.918228067$;
   $|f(1.4) - P_2(1.4)| = 0.0328284496$

   (b) $P_1(x) = 0.6099204008x - 0.1324399760$; $P_1(1.4) = 0.7214485851$;
   $|f(1.4) - P_1(1.4)| = 0.0153577147$;
   $P_2(x) = -3.183202832x^2 + 9.682048472x - 6.498845640$; $P_2(1.4) = 0.816944669$;
   $|f(1.4) - P_2(1.4)| = 0.0801383692$

   (c) $P_1(x) = 0.4012882937x - 0.0622776733$; $P_1(1.4) = 0.4995259379$;
   $|f(1.4) - P_1(1.4)| = 0.0056240404$;
   $P_2(x) = -0.2532041643x^2 + 1.122920162x - 0.5686860021$; $P_2(1.4) = 0.5071220629$;
   $|f(1.4) - P_2(1.4)| = 0.0019720846$

(d) $P_1(x) = 34.28581783x - 31.92477833$; $P_1(1.4) = 16.07536663$;
$|f(1.4) - P_1(1.4)| = 1.03071986$;
$P_2(x) = 26.85344400x^2 - 42.24649756x + 21.78210966$; $P_2(1.4) = 15.26976332$;
$|f(1.4) - P_2(1.4)| = 0.22511655$

3. Error bounds for the polynomials in Exercise 1 are as follows.

   (a) For $P_1(x)$: $\left|\frac{f''(\xi)}{2}(0.45-0)(0.45-0.6)\right| \le 0.135$;
   For $P_2(x)$: $\left|\frac{f'''(\xi)}{6}(0.45-0)(0.45-0.6)(0.45-0.9)\right| \le 0.00397$

   (b) For $P_1(x)$: $\left|\frac{f''(\xi)}{2}(0.45-0)(0.45-0.6)\right| \le 0.03375$;
   For $P_2(x)$: $\left|\frac{f'''(\xi)}{6}(0.45-0)(0.45-0.6)(0.45-0.9)\right| \le 0.001898$

   (c) For $P_1(x)$: $\left|\frac{f''(\xi)}{2}(0.45-0)(0.45-0.6)\right| \le 0.135$;
   For $P_2(x)$: $\left|\frac{f'''(\xi)}{6}(0.45-0)(0.45-0.6)(0.45-0.9)\right| \le 0.010125$

   (d) For $P_1(x)$: $\left|\frac{f''(\xi)}{2}(0.45-0)(0.45-0.6)\right| \le 0.06779$;
   For $P_2(x)$: $\left|\frac{f'''(\xi)}{6}(0.45-0)(0.45-0.6)(0.45-0.9)\right| \le 0.151$

4. Error bounds for the polynomials in Exercise 2 are as follows.

   (a) For $P_1(x)$: 0.1480440661; For $P_2(x)$: 0.2170439368

   (b) For $P_1(x)$: 0.03359789466; There is no bound since the derivative goes to ∞.

   (c) For $P_1(x)$: 0.004169227026; For $P_2(x)$: 0.006080122747

   (d) For $P_1(x)$: 1.471951812; For $P_2(x)$: 1.373821691

5. Interpolation polynomials give the following results.

(a)

n	$x_0, x_1, \ldots, x_n$	$P_n(8.4)$
1	8.3, 8.6	17.87833
2	8.3, 8.6, 8.7	17.87716
3	8.3, 8.6, 8.7, 8.1	17.87714

(b)

n	$x_0, x_1, \ldots, x_n$	$P_n(-1/3)$
1	−0.5, −0.25	0.21504167
2	−0.5, −0.25, 0.0	0.16988889
3	−0.5, −0.25, 0.0, −0.75	0.17451852

(c)

n	$x_0, x_1, \ldots, x_n$	$P_n(0.25)$
1	0.2, 0.3	−0.13869287
2	0.2, 0.3, 0.4	−0.13259734
3	0.2, 0.3, 0.4, 0.1	−0.13277477

(d)

n	$x_0, x_1, \ldots, x_n$	$P_n(0.9)$
1	0.8, 1.0	0.44086280
2	0.8, 1.0, 0.7	0.43841352
3	0.8, 1.0, 0.7, 0.6	0.44198500

6. Interpolation polynomials give the following results.

   (a) $P_1(x) = 4.278240000x + 0.579160000; P_1(0.43) = 2.418803200$
   $|f(0.43) - P_1(0.43)| = 0.055642506;$
   $P_2(x) = 5.550800000x^2 + 0.115140000x + 1.273010000;\ P_2(0.43) = 2.348863120;$
   $|f(0.43) - P_2(0.43)| = 0.014297574$
   $P_3(x) = 2.912106668x^3 + 1.182639999x^2 + 2.117213334x + 1.0;\ P_3(0.43) = 2.360604734;$
   $|f(0.43) - P_3(0.43)| = 0.002555960e$

   (b) $P_1(x) = -1.062498000x + 1.066405500; P_1(0.0) = 1.066405500$
   $|f(0.0) - P_1(0.0)| = 0.066405500;$
   $P_2(x) = 1.812509334x^2 - 1.062497999x + 0.9531236670;\ P_2(0.0) = 0.9531236670;$
   $|f(0.0) - P_2(0.0)| = 0.0468763330$
   $P_3(x) = -1.000010667x^3 + 1.312504000x^2 - 0.9999973330x + 0.9843740000;\ P_3(0.0) = 0.9843740000;$
   $|f(0.0) - P_3(0.0)| = 0.0156260000$

   (c) $P_1(x) = -2.7074748x - 0.01930238; P_1(0.18) = -0.506647844$
   $|f(0.18) - P_1(0.18)| = 0.0014756204;$
   $P_2(x) = 0.8762550000x^2 - 2.970351300x - 0.0017772800;\ P_2(0.18) = -0.5080498520;$
   $|f(0.18) - P_2(0.18)| = 0.0000736124$
   $P_3(x) = -0.4855333334x^3 + 1.167575000x^2 - 3.023759967x + 0.0011359200;\ P_3(0.18) = -0.5081430745;$
   $|f(0.18) - P_3(0.18)| = 0.0000196101$

   (d) $P_1(x) = 0.3915288000x + 1.0986123; P_1(0.25) = 1.196494500$
   $|f(0.25) - P_1(0.25)| = 0.007424569;$
   $P_2(x) = 0.1103443800x^2 + 0.3363566100x + 1.098612300;\ P_2(0.25) = 1.189597976;$
   $|f(0.25) - P_2(0.25)| = 0.000528045$
   $P_3(x) = 0.01414036000x^3 + 0.1103443800x^2 + 0.3328215200x + 1.098612300;\ P_3(0.25) = 1.188935147;$
   $|f(0.25) - P_3(0.25)| = 0.000134784$

7. The approximations are the same as in Exercise 5.

8. The approximations are the same as in Exercise 6.

9. (a) $P_2(x) = -11.22388889x^2 + 3.810500000x + 1$, and an error bound is 0.11371294.

   (b) $P_2(x) = -0.1306344167x^2 + 0.8969979335x - 0.63249693$, and an error bound is 9.45762×10^{-4}.

   (c) $P_3(x) = 0.1970056667x^3 - 1.06259055x^2 + 2.532453189x - 1.666868305$, and an error bound is 10^{-4}.

   (d) $P_3(x) = -0.07932x^3 - 0.545506x^2 + 1.0065992x + 1$, and an error bound is 1.591376×10^{-3}.

10. Error bounds when $n = 1$ and $n = 2$ are as follows.

    (a) 0.06850070205 and 0.02409356045 (b) 0.2656250000 and 0.09375000000

(c) 0.001552099938 and 0.0001109161632 (d) 0.007740087700 and 0.0007457301283

11. (a) We have $\sqrt{3} \approx P_4(1/2) = 1.708\overline{3}$. (b) We have $\sqrt{3} \approx P_4(3) = 1.690607$.

    (c) Absolute error in part (a) is approximately 0.0237, and the absolute error in part (b) is 0.0414, so part (a) is more accurate.

12. The largest value is $x_1 = 0.872677996$.

13. We have $y = 1.25$.

14. The approximation is $\cos 0.75 \approx 0.7313$. The actual error is 0.0004, and an error bound is 2.7×10^{-8}. The discrepancy is due to the fact that the data are given only to four decimal places and four digit arithmetic is used.

15. We have $f(1.09) \approx 0.2826$. The actual error is 4.3×10^{-5}, and an error bound is 7.4×10^{-6}. The discrepancy is due to the fact that the data are given to only four decimal places, and only four-digit arithmetic is used.

16. Using 10 digits gives $P_3(x) = 1.302637066x^3 - 3.511333118x^2 + 4.071141936x - 1.670043560$, $P_3(1.09) = 0.282639050$, and $|f(1.09) - P_3(1.09)| = 3.8646 \times 10^{-6}$.

17. $P_2 = f(0.7) = 6.4$

18. $P_2 = f(0.5) = 4$

19. (a) $P_2(x) = -11.22388889x^2 + 3.810500000x + 1$.
    An error bound is 0.11371294.

    (b) $P_2(x) = -0.1306344167x^2 + 0.8969979335x - 0.63249693$.
    An error bound is 9.45762×10^{-4}.

    (c) $P_3(x) = 0.1970056667x^3 - 1.06259055x^2 + 2.532453189x - 1.666868305$.
    An error bound is 10^{-4}.

    (d) $P_3(x) = -0.07932x^3 - 0.545506x^2 + 1.0065992x + 1$.
    An error bound is 1.591376×10^{-3}.

20. (a) 1.32436 (b) 2.18350 (c) $1.15277, 2.01191$

    (c) Parts (a) and (b) are better due to the spacing of the nodes.

21. The largest possible step size is 0.004291932, so 0.004 would be a reasonable choice.

22. $P_{0,1,2,3}(1.5) = 3.625$

23. $P_{0,1,2,3}(2.5) = 2.875$

24. The difference between the actual value and the computed value is $\frac{2}{3}$.

25. The first ten terms of the sequence are 0.038462, 0.333671, 0.116605, −0.371760, −0.0548919, 0.605935, 0.190249, −0.513353, −0.0668173, and 0.448335. Since $f(1+\sqrt{10}) = 0.0545716$, the sequence does not appear to converge.

26. The solution is approximately 0.567142.

27. Change Algorithm 3.1 as follows:

    INPUT numbers $y_0, y_1, ..., y_n$; values $x_0, x_1, ..., x_n$ as the first column $Q_{0,0}, Q_{1,0}, ..., Q_{n,0}$ of Q.
    OUTPUT the table Q with $Q_{n,n}$ approximating $f^{-1}(0)$.

    STEP 1 For $i = 1, 2, ..., n$
    for $j = 1, 2, ..., i$
    set

$$Q_{i,j} = \frac{y_i Q_{i-1,j-1} - y_{i-j} Q_{i,j-1}}{y_i - y_{i-j}}.$$

28. (a) $P(1930) = 169{,}649{,}000$, $P(1965) = 191{,}767{,}000$, $P(2010) = 171{,}351{,}000$
    (b) The 1965 figure may not be very accurate, but the 2010 figure is likely to be extremely inaccurate.

29. (a) Sample 1: $P_6(x) = 6.67 - 42.6434x + 16.1427x^2 - 2.09464x^3 + 0.126902x^4 - 0.00367168x^5 + 0.0000409458x^6$;
    Sample 2: $P_6(x) = 6.67 - 5.67821x + 2.91281x^2 - 0.413799x^3 + 0.0258413x^4 - 0.000752546x^5 + 0.00000836160x^6$
    (b) Sample 1: 42.71 mg; Sample 2: 19.42 mg

30. (a)

x	erf(x)
0.0	0
0.2	0.2227
0.4	0.4284
0.6	0.6039
0.8	0.7421
1.0	0.8427

    (b) Linear interpolation with $x_0 = 0.2$ and $x_1 = 0.4$ gives $\text{erf}(\frac{1}{3}) \approx 0.3598$, and quadratic interpolation with $x_0 = 0.2, x_1 = 0.4$, and $x_2 = 0.6$ gives $\text{erf}(\frac{1}{3}) \approx 0.3632$. Since $\text{erf}(\frac{1}{3}) \approx 0.3626$, quadratic interpolation is more accurate.

31. Since $g(x) = g(x_0) = 0$, there exists a number ξ_1 between x and x_0, for which $g'(\xi_1) = 0$. Also, $g'(x_0) = 0$, so there exists a number ξ_2 between x_0 and ξ_1, for which $g''(\xi_2) = 0$. The process is continued by induction to show that a number ξ_{n+1} between x_0 and ξ_n exists with $g^{(n+1)}(\xi_{n+1}) = 0$. The error formula for Taylor polynomials follows.

32. Since $g'\left(\left(j + \frac{1}{2}\right) h\right) = 0$,

$$\max |g(x)| = \max\left\{|g(jh)|, \left|g\left(\left(j + \frac{1}{2}\right)h\right)\right|, |g((j+1)h)|\right\} = \max\left(0, \frac{h^2}{4}\right),$$

    so $|g(x)| \leq h^2/4$.

33. (a) (i) $B_3(x) = x$ (ii) $B_3(x) = 1$ (d) $n \geq 250{,}000$

Exercise Set 3.2, page 127

1. The interpolating polynomials are as follows.

   (a) $P_1(x) = 16.9441 + 3.1041(x - 8.1); P_1(8.4) = 17.87533$
   $P_2(x) = P_1(x) + 0.06(x - 8.1)(x - 8.3); P_2(8.4) = 17.87713$
   $P_3(x) = P_2(x) + -0.00208333(x - 8.1)(x - 8.3)(x - 8.6); P_3(8.4) = 17.87714$

   (b) $P_1(x) = -0.1769446 + 1.9069687(x - 0.6); P_1(0.9) = 0.395146$
   $P_2(x) = P_1(x) + 0.959224(x - 0.6)(x - 0.7); P_2(0.9) = 0.4526995$
   $P_3(x) = P_2(x) - 1.785741(x - 0.6)(x - 0.7)(x - 0.8); P_3(0.9) = 0.4419850$

2. The interpolating polynomials are as follows.

   (a) $P_1(x) = 1.0 + 2.594880000x; P_1(0.43) = 2.115798400$
   $P_2(x) = P_1(x) + 3.366720000x(x - 0.25); P_2(0.43) = 2.376382528$
   $P_3(x) = P_2(x) + 2.912106667x(x - 0.25)(x - 0.5); P_3(0.43) = 2.360604734$

   (b) $P_1(x) = 0.726560000 - 2.421880000x; P_1(0) = 0.726560000$
   $P_2(x) = P_1(x) + 1.812509333(x + 0.5)(x + 0.25); P_2(0) = 0.9531236666$
   $P_3(x) = P_2(x) - 1.000010666(x + 0.5)(x + 0.25)(x - 0.25); P_3(0) = 0.9843739999$

3. In the following equations, we have $s = (1/h)(x - x_0)$.

   (a) $P_1(s) = -0.718125 - 0.0470625s;\ P_1\left(-\frac{1}{3}\right) = -0.006625$
   $P_2(s) = P_1(s) + 0.312625s(s - 1)/2;\ P_2\left(-\frac{1}{3}\right) = 0.1803056$
   $P_3(s) = P_2(s) + 0.09375s(s - 1)(s - 2)/6;\ P_3\left(-\frac{1}{3}\right) = 0.1745185$

   (b) $P_1(s) = -0.62049958 + 0.3365129s;\ P_1(0.25) = -0.1157302$
   $P_2(s) = P_1(s) - 0.04592527s(s - 1)/2;\ P_2(0.25) = -0.1329522$
   $P_3(s) = P_2(s) - 0.00283891s(s - 1)(s - 2)/6;\ P_3(0.25) = -0.1327748$

4. In the following equations, we have $s = (1/h)(x - x_0)$.

   (a) $P_1(s) = 1.0 + 0.6487200000s;\ P_1(0.43) = 2.115798400$
   $P_2(s) = P_1(s) + 0.2104200000s(s - 1);\ P_2(0.43) = 2.376382528$
   $P_3(s) = P_2(s) + 0.04550166667s(s - 1)(s - 2);\ P_3(0.43) = 2.360604734$

   (b) $P_1(s) = -0.29004986 - 0.2707474800s;\ P_1(0.18) = -0.5066478440$
   $P_2(s) = P_1(s) + 0.008762550000s(s - 1);\ P_2(0.18) = -0.5080498520$
   $P_3(s) = P_2(s) - 0.0004855333333s(s - 1)(s - 2);\ P_3(0.18) = -0.5081430744$

5. In the following equations, we have $s = (1/h)(x - x_n)$.

   (a) $P_1(s) = 1.101 + 0.7660625s;\quad f(-\frac{1}{3}) \approx P_1(-\frac{4}{3}) = 0.07958333$ $P_2(s) = P_1(s) + 0.406375s(s+1)/2;\quad f(-\frac{1}{3}) \approx P_2(-\frac{4}{3}) = 0.1698889$ $P_3(s) = P_2(s) + 0.09375s(s+1)(s+2)/6;\quad f(-\frac{1}{3}) \approx P_3(-\frac{4}{3}) = 0.1745185$