where $\xi(x)$ is between x and p. Since $f^{(m)}$ is continuous, let

$$q(x) = \frac{f^{(m)}(\xi(x))}{m!}.$$

Then $f(x) = (x-p)^m q(x)$ and

$$\lim_{x\to p} q(x) = \frac{f^{(m)}(p)}{m!} \neq 0.$$

13. If

$$\frac{|p_{n+1}-p|}{|p_n - p|^3} = 0.75 \quad \text{and} \quad |p_0 - p| = 0.5,$$

then

$$|p_n - p| = (0.75)^{(3^n-1)/2}|p_0 - p|^{3^n}.$$

To have $|p_n - p| \le 10^{-8}$ requires that $n \ge 3$.

14. Let $e_n = p_n - p$. If

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^\alpha} = \lambda > 0,$$

then for sufficiently large values of n, $|e_{n+1}| \approx \lambda |e_n|^\alpha$. Thus,

$$|e_n| \approx \lambda |e_{n-1}|^\alpha \quad \text{and} \quad |e_{n-1}| \approx \lambda^{-1/\alpha}|e_n|^{1/\alpha}.$$

Using the hypothesis gives

$$\lambda|e_n|^\alpha \approx |e_{n+1}| \approx C|e_n|\lambda^{-1/\alpha}|e_n|^{1/\alpha},$$

so

$$|e_n|^\alpha \approx C\lambda^{-1/\alpha-1}|e_n|^{1+1/\alpha}.$$

Since the powers of $|e_n|$ must agree,

$$\alpha = 1 + 1/\alpha \quad \text{and} \quad \alpha = \frac{1+\sqrt{5}}{2} \approx 1.62.$$

The number α is the *golden ratio* that appeared in Exercise 16 of section 1.3.

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1. The results are listed in the following table.

	(a)	(b)	(c)	(d)
$\hat{p}_0$	0.258684	0.907859	0.548101	0.731385
$\hat{p}_1$	0.257613	0.909568	0.547915	0.736087
$\hat{p}_2$	0.257536	0.909917	0.547847	0.737653
$\hat{p}_3$	0.257531	0.909989	0.547823	0.738469
$\hat{p}_4$	0.257530	0.910004	0.547814	0.738798
$\hat{p}_5$	0.257530	0.910007	0.547810	0.738958

2. Newton's Method gives $p_6 = -0.1828876$ and $\hat{p}_6 = -0.183387$.

3. Steffensen's method gives $p_0^{(1)} = 0.826427$.

4. Steffensen's method gives $p_0^{(1)} = 2.152905$ and $p_0^{(2)} = 1.873464$.

5. Steffensen's method gives $p_1^{(0)} = 1.5$.

6. Steffensen's method gives $p_2^{(0)} = 1.73205$.

7. For $g(x) = \sqrt{1 + \frac{1}{x}}$ and $p_0 = 1$, we have $p_3 = 1.32472$.

8. For $g(x) = 2^{-x}$ and $p_0 = 1$, we have $p_3 = 0.64119$.

9. For $g(x) = 0.5(x + \frac{3}{x})$ and $p_0 = 0.5$, we have $p_4 = 1.73205$.

10. For $g(x) = \frac{5}{\sqrt{x}}$ and $p_0 = 2.5$, we have $p_3 = 2.92401774$.

11. (a) For $g(x) = \left(2 - e^x + x^2\right)/3$ and $p_0 = 0$, we have $p_3 = 0.257530$.
    (b) For $g(x) = 0.5(\sin x + \cos x)$ and $p_0 = 0$, we have $p_4 = 0.704812$.
    (c) With $p_0 = 0.25$, $p_4 = 0.910007572$.
    (d) With $p_0 = 0.3$, $p_4 = 0.469621923$.

12. (a) For $g(x) = 2 + \sin x$ and $p_0 = 2$, we have $p_4 = 2.55419595$.
    (b) For $g(x) = \sqrt[3]{2x + 5}$ and $p_0 = 2$, we have $p_2 = 2.09455148$.
    (c) With $g(x) = \sqrt{\frac{e^x}{3}}$ and $p_0 = 1$, we have $p_3 = 0.910007574$.
    (d) With $g(x) = \cos x$, and $p_0 = 0$, we have $p_4 = 0.739085133$.

13. Aitken's Δ^2 method gives:

    (a) $\hat{p}_{10} = 0.0\overline{45}$

    (b) $\hat{p}_2 = 0.0363$

14. (a) A positive constant λ exists with

$$\lambda = \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha}.$$

Hence,

$$\lim_{n\to\infty} \left| \frac{p_{n+1} - p}{p_n - p} \right| = \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} \cdot |p_n - p|^{\alpha - 1} = \lambda \cdot 0 = 0$$

and

$$\lim_{n\to\infty} \frac{p_{n+1} - p}{p_n - p} = 0.$$

(b) One example is $p_n = \frac{1}{n^n}$.

15. We have

$$\frac{|p_{n+1}-p_n|}{|p_n-p|}=\frac{|p_{n+1}-p+p-p_n|}{|p_n-p|}=\left|\frac{p_{n+1}-p}{p_n-p}-1\right|,$$

so

$$\lim_{n\to\infty}\frac{|p_{n+1}-p_n|}{|p_n-p|}=\lim_{n\to\infty}\left|\frac{p_{n+1}-p}{p_n-p}-1\right|=1.$$

16.

$$\frac{\hat{p}_n-p}{p_n-p}=\frac{\lambda\left(\delta_n+\delta_{n+1}\right)-2\delta_n+\delta_n\delta_{n+1}-2\delta_n(\lambda-1)-\delta_n^2}{(\lambda-1)^2+\lambda\left(\delta_n+\delta_{n+1}\right)-2\delta_n+\delta_n\delta_{n+1}}$$

17. (a) First use the Taylor series for e^x to show that

$$p_n-p=-\frac{1}{(n+1)!}e^{\xi}x^{n+1},$$

where ξ is between 0 and 1. This implies that for large values of n we have

$$\left|\frac{p_{n+1}-p}{p_n-p}\right|=\left|\frac{e^{(\xi_1-\xi_2)}}{n+2}x\right|\leq 1.$$

(b)

n	p_n	$\hat{p}_n$
0	1	3
1	2	2.75
2	2.5	$2.7\overline{2}$
3	$2.\overline{6}$	2.71875
4	$2.708\overline{3}$	$2.718\overline{3}$
5	$2.71\overline{6}$	2.7182870
6	$2.7180\overline{5}$	2.7182823
7	2.7182539	2.7182818
8	2.7182787	2.7182818
9	2.7182815	
10	2.7182818	

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1. (a) For $p_0=1$, we have $p_{22}=2.69065$.

   (b) For $p_0=1$, we have $p_5=0.53209$; for $p_0=-1$, we have $p_3=-0.65270$; and for $p_0=-3$, we have $p_3=-2.87939$.

   (c) For $p_0=1$, we have $p_5=1.32472$.

   (d) For $p_0=1$, we have $p_4=1.12412$; and for $p_0=0$, we have $p_8=-0.87605$.

   (e) For $p_0=0$, we have $p_6=-0.47006$; for $p_0=-1$, we have $p_4=-0.88533$; and for $p_0=-3$, we have $p_4=-2.64561$.