

Hw 06

$$2. (i) a_n = \frac{1}{n} \Rightarrow \hat{p}_n = \frac{1}{n} - \frac{\left(\frac{1}{n+1} - \frac{1}{n}\right)^2}{\frac{1}{n+2} - \frac{2}{n+1} + \frac{1}{n}}$$

$$= \frac{1}{n} - \frac{\left(\frac{-1}{n(n+1)}\right)^2 (n+2)(n+1)n}{n(n+1) - 2n(n+2) + (n+1)(n+2)} = \frac{1}{n} - \frac{\frac{n+2}{n(n+1)}}{2}$$

$$= \frac{2n+2 - (n+2)}{2n(n+1)} = \frac{1}{2(n+1)}$$

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1} - 0|}{|a_n - 0|} = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \Rightarrow a_n \text{ converges to } 0 \text{ linearly}$$

$$\lim_{n \rightarrow \infty} \frac{|\hat{p}_{n+1} - 0|}{|\hat{p}_n - 0|} = \lim_{n \rightarrow \infty} \frac{n+1}{n+2} = 1 \Rightarrow \hat{p}_n \text{ converges to } 0 \text{ linearly}$$

$$\lim_{n \rightarrow \infty} \frac{|\hat{p}_n - 0|}{|a_n - 0|} = \frac{1}{2} \quad (\hat{p}_n \text{ faster than } a_n)$$

$$(ii) b_n = \frac{1}{n^2} \Rightarrow \hat{p}_n = \frac{1}{n^2} - \frac{\left(\frac{1}{(n+1)^2} - \frac{1}{n^2}\right)^2}{\frac{1}{(n+2)^2} - \frac{2}{(n+1)^2} + \frac{1}{n^2}} = \frac{1}{n^2} - \frac{\left(\frac{-2n-1}{n^2(n+1)^2}\right)^2 (n+2)^2 (n+1)^2 n^2}{n^2(n+1)^2 - 2n^2(n+2)^2 + (n+1)^2(n+2)^2}$$

$$= \frac{1}{n^2} - \frac{(2n+1)^2 (n+2)^2}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{(2n+1)^2 (n+2)^2}{2n^2(n+1)^2(3n^2+6n+2)}$$

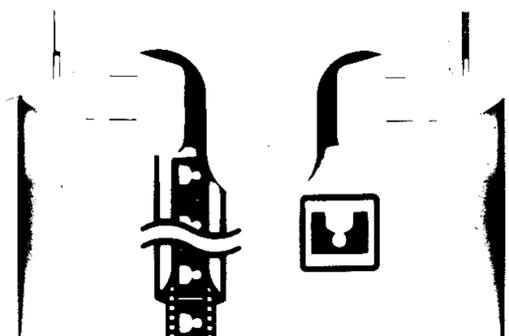
$$= \frac{2n^4 + 4n^3 + n^2}{2n^2(n+1)^2(3n^2+6n+2)} = \frac{2n^2 + 4n + 1}{2(n+1)^2(3n^2+6n+2)}$$

$$\lim_{n \rightarrow \infty} \frac{|b_{n+1} - 0|}{|b_n - 0|} = \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} = 1 \Rightarrow b_n \text{ converges to } 0 \text{ linearly}$$

$$\lim_{n \rightarrow \infty} \frac{|\hat{p}_{n+1} - 0|}{|\hat{p}_n - 0|} = \lim_{n \rightarrow \infty} \frac{\frac{2(n+1)^2 + 4(n+1) + 1}{2(n+2)^2(3(n+1)^2 + 6(n+1) + 2)}}{\frac{2n^2 + 4n + 1}{2(n+1)^2(3n^2 + 6n + 2)}} = \lim_{n \rightarrow \infty} \frac{[2(n+1)^2 + 4(n+1) + 1] 2(n+1)^2(3n^2 + 6n + 2)}{[2n^2 + 4n + 1] 2(n+2)^2(3(n+1)^2 + 6(n+1) + 2)} = 1$$

$$\Rightarrow \hat{p}_n \text{ converges to } 0 \text{ linearly}$$

$$\lim_{n \rightarrow \infty} \frac{|\hat{p}_n - 0|}{|b_n - 0|} = \frac{1}{3} \quad (\hat{p}_n \text{ faster than } b_n)$$



$$(iii) C_n = \alpha^n \Rightarrow \hat{P}_n = \alpha^n - \frac{(\alpha^{n+1} - \alpha^n)^2}{\alpha^{n+2} - 2\alpha^{n+1} + \alpha^n} = \frac{\alpha^{n+2} - 2\alpha^{n+1} + \alpha^n - (\alpha^{n+1} - \alpha^n)^2}{\alpha^{n+2} - 2\alpha^{n+1} + \alpha^n} = 0$$

$$\lim_{n \rightarrow \infty} \frac{|C_{n+1} - 0|}{|C_n - 0|} = \lim_{n \rightarrow \infty} \alpha = \alpha \Rightarrow C_n \text{ converges to } 0 \text{ linearly}$$

( $\hat{P}_n$  is much more rapidly than  $C_n$ )

$$(iv) d_n = 2^{-n} \Rightarrow \hat{P}_n = 2^{-n} - \frac{(2^{-n+1} - 2^{-n})^2}{2^{-n+2} - 2 \cdot 2^{-n+1} + 2^{-n}} = \frac{2^{-n+2} - 2 \cdot 2^{-n+1} + 2^{-n} - (2^{-n+1} - 2^{-n})^2}{2^{-n+2} - 2 \cdot 2^{-n+1} + 2^{-n}}$$

$$= \frac{(2^{-n})^5 - (2^{-n})^4}{(2^{-n})^4 - 2(2^{-n})^3 + 2^{-n}} = \frac{(2^{-n})^4 - (2^{-n})^3}{(2^{-n})^3 - 2 \cdot (2^{-n})^2 + 1}$$

$$\lim_{n \rightarrow \infty} \frac{|d_{n+1} - 0|}{|d_n - 0|^2} = \lim_{n \rightarrow \infty} \frac{2^{-n-1}}{(2^{-n})^2} = 1 \Rightarrow d_n \text{ converges to } 0 \text{ quadratically}$$

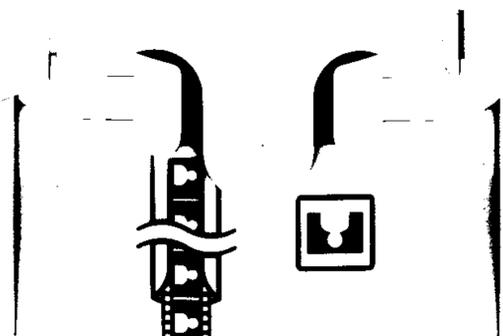
$$\lim_{n \rightarrow \infty} \frac{|\hat{P}_{n+1} - 0|}{|\hat{P}_n - 0|^2} = \lim_{n \rightarrow \infty} \frac{[(2^{-n+1})^4 - (2^{-n+1})^3] [(2^{-n})^3 - 2 \cdot 2^{-n} + 1]^2}{[(2^{-n+1})^3 - 2 \cdot 2^{-n+1} + 1] [(2^{-n})^4 - (2^{-n})^3]^2}$$

$$= \lim_{n \rightarrow \infty} \frac{[(2^{-n})^3 - 2 \cdot 2^{-n} + 1]^2 [(2^{-n})^8 - (2^{-n})^6]}{[(2^{-n+1})^3 - 2 \cdot 2^{-n+1} + 1] [(2^{-n})^8 - 2 \cdot (2^{-n})^7 + (2^{-n})^6]}$$

$$= \lim_{n \rightarrow \infty} \left| 1 \cdot \frac{[(2^{-n})^2 - 1]}{[(2^{-n})^2 - 2 \cdot 2^{-n} + 1]} \right| = 1$$

$\Rightarrow \hat{P}_n$  converges to 0 quadratically

$$\lim_{n \rightarrow \infty} \frac{|\hat{P}_n - 0|}{|d_n - 0|} = 0 \quad (\hat{P}_n \text{ is much more rapidly than } d_n)$$



$$4. (a) P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

$$\text{given } x_k, Q(x) = b_4x^3 + b_3x^2 + b_2x + b_1 \text{ s.t. } P(x) = (x - x_k)Q(x) + b_0$$

$$\text{where } b_4 = a_4 \text{ and } b_n = b_{n+1}x_k + a_n, n = 3, 2, 1, 0$$

$$P(x_k) = ( [ (a_4x_k + a_3)x_k + a_2 ] x_k + a_1 ) x_k + a_0$$

\*   Δ   \*   Δ   \*   Δ   \*   Δ

\* means multiplication : 4 次 (operation count for  $P(x_k)$ )  
 Δ means addition : 4 次

In fact, when we calculate  $P(x_k)$ , we get  $b_0 \sim b_4$  too.

$$P(x_k) = ( [ (a_4x_k + a_3)x_k + a_2 ] x_k + a_1 ) x_k + a_0$$

$\underbrace{\hspace{10em}}_{b_3}$   
 $\underbrace{\hspace{15em}}_{b_2 = b_3x_k + a_2}$   
 $\underbrace{\hspace{20em}}_{b_1 = b_2x_k + a_1}$   
 $\underbrace{\hspace{25em}}_{b_0 = b_1x_k + a_0}$

Hence, we can calculate  $P(x_k)$  and  $Q(x_k)$  (which is  $P'(x_k)$ ) in the same time

$$Q(x_k) = [ (b_4x_k + b_3)x_k + b_2 ] x_k + b_1$$

\*   Δ   \*   Δ   \*   Δ

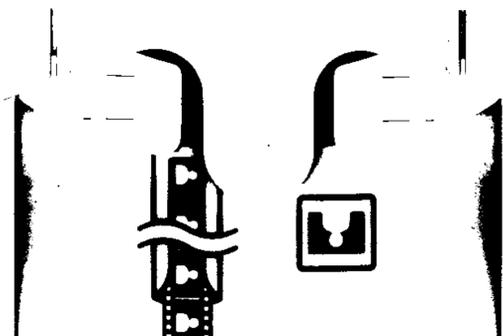
\* : 3 次

Δ : 3 次

For each Newton's iteration  $x_{k+1} = x_k - \frac{P(x_k)}{P'(x_k)}$

$$* : 4 + 3 + 1 = 8$$

$$\Delta : 4 + 3 + 1 = 8$$



b. Solving  $f(x) = e^x - 1 = 0$  by Secant method with  $0 < x_0 < x_1$

(i)

$$\text{Consider } x_2 = x_1 - f(x_1) \frac{x_1 - x_0}{f(x_1) - f(x_0)} = x_0 - f(x_0) \frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

Since  $f(x) = e^x > 0, \forall x \in \mathbb{R} \Rightarrow f$  is strictly increasing on  $\mathbb{R}$

$$\Rightarrow \frac{x_1 - x_0}{f(x_1) - f(x_0)} > 0$$

$$\text{and } f(0) = 0 \Rightarrow f(x_1) > f(x_0) > 0 \Rightarrow x_2 < x_0 < x_1$$

(ii)

Since  $f''(x) = e^x > 0, \forall x \in \mathbb{R} \Rightarrow f$  is convex upward on  $\mathbb{R}$

$$\Rightarrow \forall x, y \in \mathbb{R}, f(dx + (1-d)y) \leq df(x) + (1-d)f(y), 0 \leq d \leq 1$$

$$\text{If } x_2 < 0 \Rightarrow f(x_2) < f(0) = 0$$

$$\text{Since } \frac{-x_2}{x_0 - x_2} > 0 \text{ and } \frac{x_0}{x_0 - x_2} > 0 \text{ and } \frac{-x_2}{x_0 - x_2} + \frac{x_0}{x_0 - x_2} = 1$$

$$\Rightarrow f\left(\frac{-x_2}{x_0 - x_2}x_0 + \frac{x_0}{x_0 - x_2}x_2\right) \leq \frac{-x_2}{x_0 - x_2}f(x_0) + \frac{x_0}{x_0 - x_2}f(x_2) = \frac{x_0 f(x_2) - x_2 f(x_0)}{x_0 - x_2}$$

$$\parallel \\ 0 = f(0)$$

$$< \frac{x_0 f(x_2) - x_2 f(x_0)}{x_0 - x_2} = f(x_2) < 0$$

(contradiction)

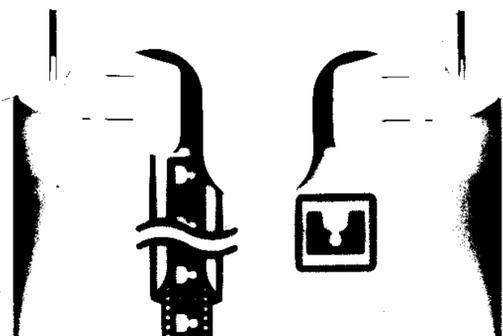
Hence  $x_2 \geq 0$ , similarly we can get  $x_1 > x_0 > x_2 > x_3 > \dots > x_k \geq 0$

where  $x_1, x_0, x_2, x_3, \dots$  is strictly decreasing and bounded below by 0

$\Rightarrow$  The sequence would converge to some  $x^*$

$$\text{Since } x_{n+2} = x_{n+1} - \frac{f(x_{n+1})(x_{n+1} - x_n)}{f(x_{n+1}) - f(x_n)} \text{ and } \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n} = f'(\xi_n) \rightarrow f'(x^*) \text{ as } n \rightarrow \infty$$

where  $\xi_n$  is between  $x_n$  &  $x_{n+1}$



$\Rightarrow x^* = x^* - \frac{f(x^*)}{f'(x^*)}$ , which means  $x^*$  is a fixed point of Newton's iteration

Since 0 is the only root of  $f(x)=0$   
 $\Rightarrow$  The iteration would converge to  $x^*=0$

1.

Solving  $f(x)=e^x-1=0$  by false position with  $x_1 < 0 < x_0$

Since  $f(0)=0$  and  $f'(x)=e^x > 0, \forall x \in \mathbb{R} \Rightarrow f(x_1) < 0 < f(x_0)$

Then  $x_2 = x_1 - f(x_1) \frac{x_1 - x_0}{f(x_1) - f(x_0)} > x_1$

Also,  $x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{f(x_1)}{f(x_1) - f(x_0)} x_0 + \frac{-f(x_0)}{f(x_1) - f(x_0)} x_1$

where  $\frac{f(x_1)}{f(x_1) - f(x_0)} > 0$  &  $\frac{-f(x_0)}{f(x_1) - f(x_0)} > 0$  &  $\frac{f(x_1)}{f(x_1) - f(x_0)} + \frac{-f(x_0)}{f(x_1) - f(x_0)} = 1$

Hence  $f(x_2) \leq \frac{f(x_1)}{f(x_1) - f(x_0)} f(x_0) + \frac{-f(x_0)}{f(x_1) - f(x_0)} f(x_1) = 0 \Rightarrow x_2 \leq 0$

Similarly, we can get  $x_1 < x_2 < x_3 < \dots < x_k < \dots \leq 0 < x_0$

The sequence  $x_1, x_2, x_3, \dots$  is increasing and bounded above by 0

With the same discussion in Problem 6

$\Rightarrow$  The iteration would converge to 0

8. By the discussion in Problem 7, we know the sequence is like

$x_1 < x_2 < x_3 < \dots \leq 0 < x_0$  and converges to 0

Hence, our iteration is like  $x_{n+1} = x_n - f(x_n) \frac{x_n - x_0}{f(x_n) - f(x_0)}$ ,  $n=2, 3, \dots$

Since  $f(x_n) = f(x_0) + f'(\xi_n)(x_n - x_0)$ , where  $\xi_n$  is between  $x_n$  &  $x_0$

and  $f(x_n) = e^{x_n} - 1 = (1 + x_n + \frac{e^{\delta_n}}{2} x_n^2) - 1$ , where  $\delta_n$  is between  $x_n$  & 0

$$= x_n + \frac{e^{\delta_n}}{2} x_n^2$$

then,  $x_{n+1} = x_n - \left(x_n + \frac{e^{\delta_n}}{2} x_n^2\right) \frac{x_n - x_0}{f'(\xi_n)(x_n - x_0)} = \left(1 - \frac{1}{f'(\xi_n)}\right) x_n - \frac{e^{\delta_n}}{2 f'(\xi_n)} x_n^2$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{|x_{n+1} - 0|}{|x_n - 0|} = \lim_{n \rightarrow \infty} \left| \left(1 - \frac{1}{f'(\xi_n)}\right) - \frac{e^{\delta_n}}{2 f'(\xi_n)} x_n \right|$$

Here,  $\lim_{n \rightarrow \infty} f'(\xi_n) = \frac{f(0) - f(x_0)}{0 - x_0} = \frac{-2x_0}{-x_0} = 2$  (Since  $x_0$  is a root of  $e^x - 1 = 2x$ )

Hence  $\lim_{n \rightarrow \infty} \frac{|x_{n+1} - 0|}{|x_n - 0|} = \frac{1}{2} \Rightarrow$  The sequence converges linearly to 0

