HW06

2. (i) $a_n = \frac{1}{n} \Rightarrow \hat{p}_n = \frac{1}{n} - \dfrac{\left(\frac{1}{n+1} - \frac{1}{n}\right)^2}{\frac{1}{n+2} - \frac{2}{n+1} + \frac{1}{n}}$

$$= \frac{1}{n} - \frac{\left(\frac{-1}{n(n+1)}\right)^2 (n+2)(n+1)n}{n(n+1) - 2n(n+2) + (n+1)(n+2)} = \frac{1}{n} - \frac{\frac{n+2}{n(n+1)}}{2}$$

$$= \frac{2n+2-(n+2)}{2n(n+1)} = \frac{1}{2(n+1)}$$

$\lim\limits_{n\to\infty} \frac{|a_{n+1}-0|}{|a_n-0|} = \lim\limits_{n\to\infty} \frac{n}{n+1} = 1 \Rightarrow a_n$ converges to 0 linearly

$\lim\limits_{n\to\infty} \frac{|\hat{p}_{n+1}-0|}{|\hat{p}_n-0|} = \lim\limits_{n\to\infty} \frac{n+1}{n+2} = 1 \Rightarrow \hat{p}_n$ converges to 0 linearly

$\lim\limits_{n\to\infty} \frac{|\hat{p}_n-0|}{|a_n-0|} = \frac{1}{2}$ ($\hat{p}_n$ faster than a_n)

(ii) $b_n = \frac{1}{n^2} \Rightarrow \hat{p}_n = \frac{1}{n^2} - \dfrac{\left(\frac{1}{(n+1)^2} - \frac{1}{n^2}\right)^2}{\frac{1}{(n+2)^2} - \frac{2}{(n+1)^2} + \frac{1}{n^2}} = \frac{1}{n^2} - \dfrac{\left(\frac{-2n-1}{n^2(n+1)^2}\right)^2 (n+2)^2(n+1)^2 n^2}{n^2(n+1)^2 - 2n^2(n+2)^2 + (n+1)^2(n+2)^2}$

$$= \frac{1}{n^2} - \frac{\frac{(2n+1)^2(n+2)^2}{n^2(n+1)^2}}{6n^2+12n+4} = \frac{1}{n^2} - \frac{(2n+1)^2(n+2)^2}{2n^2(n+1)^2(3n^2+6n+2)}$$

$$= \frac{2n^4+4n^3+n^2}{2n^2(n+1)^2(3n^2+6n+2)} = \frac{2n^2+4n+1}{2(n+1)^2(3n^2+6n+2)}$$

$\lim\limits_{n\to\infty} \frac{|b_{n+1}-0|}{|b_n-0|} = \lim\limits_{n\to\infty} \frac{n^2}{(n+1)^2} = 1 \Rightarrow b_n$ converges to 0 linearly

$$\lim_{n\to\infty} \frac{|\hat{p}_{n+1}-0|}{|\hat{p}_n-0|} = \lim_{n\to\infty} \frac{\frac{2(n+1)^2+4(n+1)+1}{2(n+2)^2(3(n+1)^2+6(n+1)+2)}}{\frac{2n^2+4n+1}{2(n+1)^2(3n^2+6n+2)}} = \lim_{n\to\infty} \frac{[2(n+1)^2+4(n+1)+1]\,2(n+1)^2(3n^2+6n+2)}{[2n^2+4n+1]\,2(n+1)^2(3(n+1)^2+6(n+1)+2)} = 1$$

$\Rightarrow \hat{p}_n$ converges to 0 linearly

$\lim\limits_{n\to\infty} \frac{|\hat{p}_n-0|}{|b_n-0|} = \frac{1}{3}$ ($\hat{p}_n$ faster than b_n)

(iii) $c_n=\alpha^n \Rightarrow \hat{P}_n = \alpha^n - \frac{(\alpha^{n+1}-\alpha^n)^2}{\alpha^{n+2}-2\alpha^{n+1}+\alpha^n} = \frac{\alpha^{2n+2}-2\alpha^{2n+1}+\alpha^{2n}-\alpha^{2n+2}+2\alpha^{2n+1}-\alpha^{2n}}{\alpha^{n+2}-2\alpha^{n+1}+\alpha^n} = 0$

$\lim_{n\to\infty}\frac{|c_{n+1}-0|}{|c_n-0|} = \lim_{n\to\infty}\alpha = \alpha \Rightarrow c_n$ converges to 0 linearly

($\hat{P}_n$ is much more rapidly than c_n)

(iv) $d_n = 2^{-2^n} \Rightarrow \hat{P}_n = 2^{-2^n} - \frac{(2^{-2^{n+1}}-2^{-2^n})^2}{2^{-2^{n+2}}-2\cdot 2^{-2^{n+1}}+2^{-2^n}} = \frac{2^{-2^n-2^{n+2}}-2\cdot 2^{-2^n-2^{n+1}}+2^{-2^n-2^n}-2^{-2^{n+2}}+2\cdot 2^{-2^{n+1}-2^n}-2^{-2^{n+1}}}{2^{-2^{n+2}}-2\cdot 2^{-2^{n+1}}+2^{-2^n}}$

$$= \frac{(2^{-2^n})^5-(2^{-2^n})^4}{(2^{-2^n})^4-2(2^{-2^n})^2+2^{-2^n}} = \frac{(2^{-2^n})^4-(2^{-2^n})^3}{(2^{-2^n})^3-2\cdot(2^{-2^n})+1}$$

$\lim_{n\to\infty}\frac{|d_{n+1}-0|}{|d_n-0|^2} = \lim_{n\to\infty}\frac{2^{-2^{n+1}}}{(2^{-2^n})^2} = 1 \Rightarrow d_n$ converges to 0 quadratically

$$\lim_{n\to\infty}\frac{|\hat{P}_{n+1}-0|}{|\hat{P}_n-0|^2} = \lim_{n\to\infty}\left|\frac{[(2^{-2^{n+1}})^4-(2^{-2^{n+1}})^3]}{[(2^{-2^{n+1}})^3-2\cdot 2^{-2^{n+1}}+1]}\cdot\frac{[(2^{-2^n})^3-2\cdot 2^{-2^n}+1]^2}{[(2^{-2^n})^4-(2^{-2^n})^3]^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{[(2^{-2^n})^3-2\cdot 2^{-2^n}+1]^2}{[(2^{-2^{n+1}})^3-2\cdot 2^{-2^{n+1}}+1]}\cdot\frac{[(2^{-2^n})^8-(2^{-2^n})^6]}{[(2^{-2^n})^8-2\cdot(2^{-2^n})^7+(2^{-2^n})^6]}\right|$$

$$= \lim_{n\to\infty}\left|1\cdot\frac{[(2^{-2^n})^2-1]}{[(2^{-2^n})^2-2\cdot 2^{-2^n}+1]}\right| = 1$$

$\Rightarrow \hat{P}_n$ converges to 0 quadratically

$\lim_{n\to\infty}\frac{|\hat{P}_n-0|}{|d_n-0|} = 0$ ($\hat{P}_n$ is much more rapidly than d_n)

4. (a) $P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$

given x_k, $Q(x) = b_4x^3 + b_3x^2 + b_2x + b_1$ s.t. $P(x) = (x - x_k)Q(x) + b_0$

where $b_4 = a_4$ and $b_n = b_{n+1}x_k + a_n$, $n = 3, 2, 1, 0$

$$P(x_k) = ([(a_4x_k + a_3)x_k + a_2]x_k + a_1)x_k + a_0$$

* △ * △ * △ * △

* means multiplication : 4 次
△ means addition : 4 次 ~~(operation count for $P(x_k)$)~~

In fact, when we calculate $P(x_k)$, we get $b_0 \sim b_4$ too.

$$P(x_k) = \Big(\Big[\big(\underbrace{\overset{b_4}{\overset{\parallel}{a_4}}x_k + a_3}_{b_3}\big)x_k + a_2\Big]x_k + a_1\Big)x_k + a_0$$

$b_2 = b_3x_k + a_2$

$b_1 = b_2x_k + a_1$

$b_0 = b_1x_k + a_0$

Hence, we can calculate $P(x_k)$ and $Q(x_k)$ (which is $P'(x_k)$) in the same time

$$Q(x_k) = [(b_4x_k + b_3)x_k + b_2]x_k + b_1$$

* △ * △ * △

* : 3 次
△ : 3 次

For each Newton's iteration $x_{k+1} = x_k - \frac{P(x_k)}{P'(x_k)}$
* : $4 + 3 + 1 = 8$
△ : $4 + 3 + 1 = 8$

6. Solving $f(x)=e^x-1=0$ by Secant method with $0<x_0<x_1$

(i)

Consider $x_2=x_1-f(x_1)\frac{x_1-x_0}{f(x_1)-f(x_0)}=x_0-f(x_0)\frac{x_1-x_0}{f(x_1)-f(x_0)}$

Since $f'(x)=e^x>0$, $\forall x\in\mathbb{R} \Rightarrow f$ is strictly increasing on $\mathbb{R}$

$$\Rightarrow \frac{x_1-x_0}{f(x_1)-f(x_0)}>0$$

and $f(0)=0 \Rightarrow f(x_1)>f(x_0)>0 \Rightarrow x_2<x_0<x_1$

(ii)

Since $f''(x)=e^x>0$, $\forall x\in\mathbb{R} \Rightarrow f$ is convex upward on $\mathbb{R}$

$\Rightarrow \forall x,y\in\mathbb{R}$, $f(\alpha x+(1-\alpha)y)\le \alpha f(x)+(1-\alpha)f(y)$, $0\le\alpha\le1$

If $x_2<0 \Rightarrow f(x_2)<f(0)=0$

Since $\frac{-x_2}{x_0-x_2}>0$ and $\frac{x_0}{x_0-x_2}>0$ and $\frac{-x_2}{x_0-x_2}+\frac{x_0}{x_0-x_2}=1$

$$\Rightarrow \underbrace{f\left(\frac{-x_2}{x_0-x_2}x_0+\frac{x_0}{x_0-x_2}x_2\right)}_{0=f(0)}\le \frac{-x_2}{x_0-x_2}f(x_0)+\frac{x_0}{x_0-x_2}f(x_2)=\frac{x_0f(x_2)-x_2f(x_0)}{x_0-x_2}$$

$$<\frac{x_0f(x_2)-x_2f(x_2)}{x_0-x_2}=f(x_2)<0$$

(contradiction)

Hence $x_2\ge0$, similarly we can get $x_1>x_0>x_2>x_3>\cdots>x_k\ge0$

where $x_1, x_0, x_2, x_3, \cdots$ is strictly decreasing and bounded below by 0

$\Rightarrow$ The sequence would converge to some x^*

Since $x_{n+2}=x_{n+1}-\frac{f(x_{n+1})(x_{n+1}-x_n)}{f(x_{n+1})-f(x_n)}$ and $\frac{f(x_{n+1})-f(x_n)}{x_{n+1}-x_n}=f'(\xi_n)\to f'(x^*)$ as $n\to\infty$

where ξ_n is between x_n & x_{n+1}

$\Rightarrow x^* = x^* - \frac{f(x^*)}{f'(x^*)}$, which means x^* is a fixed point of Newton's iteration

Since 0 is the only root of $f(x)=0$

$\Rightarrow$ The iteration would converge to $x^*=0$

7.

Solving $f(x)=e^x-1=0$ by false position with $x_1<0<x_0$

Since $f(0)=0$ and $f'(x)=e^x>0$, $\forall x\in\mathbb{R} \Rightarrow f(x_1)<0<f(x_0)$

Then $x_2 = x_1 - f(x_1)\frac{x_1-x_0}{f(x_1)-f(x_0)} > x_1$

Also, $x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1)-f(x_0)} = \frac{f(x_1)}{f(x_1)-f(x_0)}x_0 + \frac{-f(x_0)}{f(x_1)-f(x_0)}x_1$

Where $\frac{f(x_1)}{f(x_1)-f(x_0)}>0$ & $\frac{-f(x_0)}{f(x_1)-f(x_0)}>0$ & $\frac{f(x_1)}{f(x_1)-f(x_0)} + \frac{-f(x_0)}{f(x_1)-f(x_0)} = 1$

Hence $f(x_2) \le \frac{f(x_1)}{f(x_1)-f(x_0)}f(x_0) + \frac{-f(x_0)}{f(x_1)-f(x_0)}f(x_1) = 0 \Rightarrow x_2 \le 0$

Similarly, we can get $x_1 < x_2 < x_3 < \cdots < x_k < \cdots \le 0 < x_0$

The sequence $x_1, x_2, x_3, \cdots$ is increasing and bounded above by 0

With the same discussion in Problem 6

$\Rightarrow$ The iteration would converge to 0

8. By the discussion in Problem 7, we know the sequence is like

$x_1 < x_2 < x_3 < \cdots \leq 0 < x_0$ and converges to 0

Hence, our iteration is like $x_{n+1} = x_n - f(x_n)\dfrac{x_n - x_0}{f(x_n) - f(x_0)}$, $n = 2, 3, \cdots$

Since $f(x_n) = f(x_0) + f'(\xi_n)(x_n - x_0)$, where ξ_n is between x_n & x_0

and $f(x_n) = e^{x_n} - 1 = (1 + x_n + \frac{e^{\delta_n}}{2}x_n^2) - 1$, where δ_n is between x_n & 0

$$= x_n + \frac{e^{\delta_n}}{2}x_n^2$$

then, $x_{n+1} = x_n - \left(x_n + \frac{e^{\delta_n}}{2}x_n^2\right)\dfrac{x_n - x_0}{f'(\xi_n)(x_n - x_0)} = \left(1 - \frac{1}{f'(\xi_n)}\right)x_n - \frac{e^{\delta_n}}{2f'(\xi_n)}x_n^2$

$$\Rightarrow \lim_{n\to\infty}\frac{|x_{n+1} - 0|}{|x_n - 0|} = \lim_{n\to\infty}\left|\left(1 - \frac{1}{f'(\xi_n)}\right) - \frac{e^{\delta_n}}{2f'(\xi_n)}x_n\right|$$

Here, $\lim_{n\to\infty} f'(\xi_n) = \frac{f(0) - f(x_0)}{0 - x_0} = \frac{-2x_0}{-x_0} = 2$ (Since x_0 is a root of $e^x - 1 = 2x$)

Hence, $\lim_{n\to\infty}\frac{|x_{n+1} - 0|}{|x_n - 0|} = \frac{1}{2}$ $\Rightarrow$ The sequence converges linearly to 0