Hw05

4. f has a zero of multiplicity m at x^*

$\Rightarrow f(x) = (x-x^*)^m q(x)$ where $q(x^*) \neq 0$

$$\Rightarrow g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{(x-x^*)^m q(x)}{m(x-x^*)^{m-1}q(x) + (x-x^*)^m q'(x)}$$

$$= x - \frac{(x-x^*)q(x)}{mq(x)+(x-x^*)q'(x)} \quad \text{when } x \neq x^*$$

$$\Rightarrow g'(x) = 1 - \frac{[q(x)+(x-x^*)q'(x)][mq(x)+(x-x^*)q'(x)] - (x-x^*)q(x)[mq(x)+(x-x^*)q'(x)]'}{[mq(x)+(x-x^*)q'(x)]^2}, \text{ when } x \neq x^*$$

If $f \in C^2[a,b] \Rightarrow g'(x) = 1 - \frac{(f'(x))^2 - f(x)\cdot f''(x)}{(f'(x))^2}$ is continuous

$$\Rightarrow g'(x^*) = \lim_{x \to x^*} g'(x) = 1 - \frac{q(x^*)\, m q(x^*)}{[mq(x^*)]^2} = 1 - \frac{1}{m}$$

For Newton's method,

if $m=1 \Rightarrow$ The iteration converges quadratically to the root

if $m \geq 2 \Rightarrow$ The iteration converges linearly to the root

When m becomes larger $\Rightarrow g'(x^*) \to 1$

$\Rightarrow$ The iteration would converge much more slowly.

5. (i) If $a \neq 0$ is another root of $f(x)=0$

$\Rightarrow \exists\, x_1$ between a & 0 s.t. $f'(x_1) = \frac{f(a)-f(0)}{a-0} = 0$

$\Rightarrow \exists\, x_2$ between x_1 & 0 s.t. $f''(x_2) = \frac{f'(x_1)-f'(0)}{x_1-0} = 0$

But $f''(x) > 0$, $\forall x \in \mathbb{R}$, so $x=0$ is the unique root of $f(x)=0$

(ii) Give any initial point $x_0 \in \mathbb{R}$, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ for $n=0,1,2,\cdots$

Since $f''(x) > 0$ for all $x \in \mathbb{R}$ $\Rightarrow$ $f'(x)$ increases for all $x \in \mathbb{R}$

$\because f'(0)=0 \Rightarrow f'(x) < 0$, $\forall x<0$ and $f'(x)>0$, $\forall x>0$
(f decreases at $x<0$) (f increases at $x>0$)

$\because f(0)=0 \Rightarrow f(x) > 0$ at $x \neq 0$

① If $x_0 > 0$

$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{x_0 f'(x_0) - f(x_0)}{f'(x_0)}$

Since $\frac{f(x_0)-f(0)}{x_0-0} = f'(c_0)$ for some c_0 between 0 and x_0 $(0<c_0<x_0)$

$\Rightarrow f(x_0) = x_0 f'(c_0) < x_0 f'(x_0)$ (f' increases for all $x \in \mathbb{R}$)

$\Rightarrow x_1 = \frac{x_0 f'(x_0) - f(x_0)}{f'(x_0)} > 0$

Moreover, $\frac{f(x_0)}{f'(x_0)} > 0 \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} < x_0$

Similarly, we have $x_0 > x_1 > x_2 > \cdots$ and $x_n > 0$, $n=0,1,2,\cdots$

Hence $\{x_n\}_{n=1}^{\infty}$ is a decreasing sequence and bounded below by 0

So $\{x_n\}_{n=1}^{\infty}$ would converge

Since $\{x_n\}_{n=1}^{\infty}$ generated by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n=0,1,2,\cdots$

$\Rightarrow$ $\{x_n\}_{n=1}^{\infty}$ would converge to the fixed point of $g(x) = x - \frac{f(x)}{f'(x)}$

Since we just proved that $x=0$ is the unique root of $f(x)=0$

$\Rightarrow$ $\{x_n\}_{n=1}^{\infty}$ would converge to 0, if $x_0 > 0$

② If $x_0 < 0$

Similar to ① $\{x_n\}_{n=1}^{\infty}$ would be a increasing sequence and bounded above by 0

$\Rightarrow$ $\{x_n\}_{n=1}^{\infty}$ would converge to 0, if $x_0 < 0$

By ①, ②, Newton's method converges globally to 0, regardless of x_0

(iii) Since $f(0) = f'(0) = 0$ and $f''(x) > 0$ for all $x \in \mathbb{R}$

$\Rightarrow$ f has a zero of multiplicity 2 at 0

$\Rightarrow$ $g'(0) = 1 - \frac{1}{2} = \frac{1}{2} \neq 0$, where $g(x) = x - \frac{f(x)}{f'(x)}$

$\Rightarrow$ The iteration converges linearly to 0

6. Since $g'(x^*)=0 \Rightarrow$ given $0<k<1$, $\exists \delta>0$ s.t. $|g'(x)|\le k<1$, $\forall x\in[x^*-\delta, x^*+\delta]$

The argument used in the proof of Theorem 2.5 (P66) shows that the terms of the sequence $\{x_n\}_{n=0}^{\infty}$ are contained in $[x^*-\delta, x^*+\delta]$

Consider Taylor expansion of $g(x)$ at x^*

$$g(x)=\underset{x^*}{\underset{\parallel}{g(x^*)}}+\underset{0}{\underset{\parallel}{g'(x^*)}}(x-x^*)+\underset{0}{\underset{\parallel}{\frac{g''(x^*)}{2}}}(x-x^*)^2+\frac{g'''(\xi)}{6}(x-x^*)^3,$$ where ξ is between x and x^*

$\Rightarrow g(x_n)=x^*+\frac{g'''(\xi_n)}{6}(x_n-x^*)^3$, where ξ_n is between x_n and x^*

$$\Rightarrow \lim_{n\to\infty}\frac{|x_{n+1}-x^*|}{|x_n-x^*|^3}=\lim_{n\to\infty}\frac{|g(x_n)-x^*|}{|x_n-x^*|^3}=\lim_{n\to\infty}\left|\frac{g'''(\xi_n)}{6}\right|=\left|\frac{g'''(x^*)}{6}\right|$$

Hence $\{x_n\}_{n=0}^{\infty}$ converges locally with cubic order