HW 04

2. Let x^0 be the initial point

and $x^{n+1} = \alpha x^n + (1-\alpha) g(x^n)$, $n = 0, 1, 2, \cdots$

If x^n converges to x and $g(x)$ is continuous on domain

then $g(x) = g(\lim_{n\to\infty} x^n) = \lim_{n\to\infty} g(x^n) = \lim_{n\to\infty} \frac{x^{n+1} - \alpha x^n}{1-\alpha} = \frac{x - \alpha x}{1-\alpha} = x$

Ex: $g(x) = 1 - 2x + 0.2 \sin x$ on $[-5, 5]$

$\Rightarrow g'(x) = -2 + 0.2 \cos x$ on $[-5, 5]$

$\Rightarrow -2 \le g'(x) \le -1.8$ on $[-5, 5]$

Since $|g'(x)| \ge 1.8 > 1$ on $[-5, 5]$

The iteration $x^{n+1} = g(x^n)$ almost for sure does not converge

Let $\tilde{g}(x) = \alpha x + (1-\alpha) g(x)$

$\Rightarrow \tilde{g}'(x) = \alpha + (1-\alpha) g'(x)$

Asking for $|\tilde{g}'(x)| < 1 \Rightarrow -1 < \alpha + (1-\alpha) g'(x) < 1$

$\Rightarrow -1 < \alpha(1 - g'(x)) + g'(x) < 1$

$\Rightarrow -1 - g'(x) < \alpha(1 - g'(x)) < 1 - g'(x)$

Since $-2 \le g'(x) \le -1.8 \Rightarrow 2.8 \le 1 - g'(x) \le 3$

Hence $\frac{-1 - g'(x)}{1 - g'(x)} < \alpha < 1$, we have to find α s.t the

inequality holds for every $x \in [-5, 5]$

Since $0.8 \le -1-g'(x) \le 1$ & $2.8 \le 1-g'(x) \le 3$

$$\Rightarrow \max_{x\in[-5,5]} \frac{-1-g'(x)}{1-g'(x)} \le \frac{1}{2.8} \doteqdot 0.3571$$

Hence, if we choose $\alpha=0.4$

and let $\tilde{g}(x) = \alpha x + (1-\alpha) g(x)$

Give $x_0 \in [-1,1]$ and $x^{n+1} = \tilde{g}(x^n)$, $n=0,1,2,\cdots$

Since $\tilde{g}(-5) \doteqdot 4.71 > -5$ and $\tilde{g}(5) \doteqdot -3.51 < 5$

Moreover $|\tilde{g}'(x)|<1$, $-5 \le g(x) \le 5$ on $[-5,5]$

Hence the iteration would converge to the fixed point of g