Final

1. (a) Let $f(x) = \sin x - \frac{1}{2}(\cos 1 - 1)x^2 \Rightarrow f'(x) = \cos x - (\cos 1 - 1)x$

$$\Rightarrow f''(x) = -\sin x - (\cos 1 - 1)$$

Let $h = \frac{1-0}{n+2} = \frac{1}{n+2} \Rightarrow \int_0^1 f(x)\,dx = 2h\sum_{i=0}^{\frac{n}{2}} f(x_{2i}) + \frac{1-0}{6}h^2 f''(\mu)$, where $\mu \in (0,1)$

Since $f'''(x) = -\cos x < 0$ on $[0,1] \Rightarrow f''(x)$ is decreasing on $[0,1]$

$$\Rightarrow \max_{0\le x\le 1} |f''(x)| = \max\{|f''(0)|, |f''(1)|\} \doteqdot 0.45969\cdots$$

To bound the error within $10^{-6} \Rightarrow \frac{1}{6}h^2 \cdot 0.46 \le 10^{-6} \Rightarrow h \le 3.611575\cdots \times 10^{-3}$

$$\Rightarrow n \ge 274.88\cdots$$

(b)

$h = 0.01 \Rightarrow M_{100} = 0.53631397657655 69$

$h = 0.005 \Rightarrow M_{200} = 0.5363139764927570$

$h = 0.0025 \Rightarrow M_{400} = 0.5363139764875197$

Order of convergence $= \log_2\left(\frac{\text{Exact} - M_{100}}{\text{Exact} - M_{200}}\right) \doteqdot 3.999976\cdots \doteqdot 4$

or

$= \log_2\left(\frac{M_{100} - M_{200}}{M_{200} - M_{400}}\right) \doteqdot 4.000034\cdots \doteqdot 4$

(c) Why is the numerical order greater than 2?

Use Taylor's expansion : $\int_{x_{2i-1}}^{x_{2i+1}} f(x)\,dx = \int_{x_{2i-1}}^{x_{2i+1}} f(x_{2i}) + f'(x_{2i})(x-x_{2i}) + \frac{f''(x_{2i})}{2}(x-x_{2i})^2 + \frac{f'''(x_{2i})}{6}(x-x_{2i})^3 + \frac{f^{(4)}(\xi)}{24}(x-x_{2i})^4\,dx$

$$= 2hf(x_{2i}) + \frac{f''(x_{2i})}{3}h^3 + \frac{f^{(4)}(\xi_{2i})}{60}h^5, \text{ where } \xi_{2i} \in (x_{2i-1}, x_{2i+1})$$

$$\Rightarrow \int_0^1 f(x)\,dx = \underbrace{2h\sum_{i=0}^{\frac{n}{2}} f(x_{2i})}_{\text{Midpoint rule}} + \frac{h^3}{3}\sum_{i=0}^{\frac{n}{2}} f''(x_{2i}) + \frac{h^5}{60}\sum_{i=0}^{\frac{n}{2}} f^{(4)}(\xi_{2i})$$

Now we consider $\frac{h^3}{3}\sum_{i=0}^{\frac{n}{2}} f''(x_{2i})$ and $\frac{h^5}{60}\sum_{i=0}^{\frac{n}{2}} f^{(4)}(\xi_{2i})$

① $2h\sum_{i=0}^{\frac{n}{2}} f''(x_{2i})$: Midpoint rule for $\int_0^1 f''(x)dx$

$$\Rightarrow \int_0^1 f''(x)dx = 2h\sum_{i=0}^{\frac{n}{2}} f''(x_{2i}) + \sum_{i=0}^{\frac{n}{2}} \frac{h^3}{3} f^{(4)}(\xi_{2i}), \text{ where } \xi_{2i} \in (x_{2i-1}, x_{2i+1})$$

$$\Rightarrow 2h\sum_{i=0}^{\frac{n}{2}} f''(x_{2i}) = \int_0^1 f''(x)dx - \sum_{i=0}^{\frac{n}{2}} \frac{h^3}{3} f^{(4)}(\xi_{2i}) \quad (\text{since } f'(1)-f'(0)=0)$$

$$= -\frac{h^2}{6}\left[2h\sum_{i=0}^{\frac{n}{2}} f^{(4)}(\xi_{2i})\right] = O(h^2) \quad (\text{since } 2h\sum_{i=0}^{\frac{n}{2}} f^{(4)}(\xi_{2i}) \longrightarrow f'''(1)-f'''(0) = 1-\cos 1 \text{ as } h \to 0)$$

② $\frac{h^5}{60}\sum_{i=0}^{\frac{n}{2}} f^{(4)}(\xi_{2i}) = \frac{h^4}{120}\left[2h\sum_{i=0}^{\frac{n}{2}} f^{(4)}(\xi_{2i})\right] = O(h^4)$ (same as above)

Hence, $\int_0^1 f(x)dx - 2h\sum_{i=0}^{\frac{n}{2}} f(x_{2i}) = \underbrace{\frac{h^2}{6}\left[2h\sum_{i=0}^{\frac{n}{2}} f''(x_{2i})\right]}_{O(h^2)} + \underbrace{\frac{h^5}{60}\sum_{i=0}^{\frac{n}{2}} f^{(4)}(\xi_{2i})}_{O(h^4)} = O(h^4)$

2. $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(\xi(x))}{24}x^4$, where $\xi(x)$ is between 0 & x

$$\int_{-1}^1 f(x)\,dx = 2f(0) + \frac{f''(0)}{3} + \frac{1}{24}\int_{-1}^1 f^{(4)}(\xi(x))x^4dx$$

By Weighted Mean Value Theorem, $\int_{-1}^1 f^{(4)}(\xi(x))x^4dx = f^{(4)}(\xi_1)\int_{-1}^1 x^4dx = \frac{2}{5}f^{(4)}(\xi_1)$, $\xi_1 \in (-1,1)$

Moreover, $f''(0) = \frac{1}{1^2}[f(-1)-2f(0)+f(1)] - \frac{1^2}{12}f^{(4)}(\xi_2)$, $\xi_2 \in (-1,1)$

$$\Rightarrow \int_{-1}^1 f(x)dx = \frac{1}{3}[f(-1)+4f(0)+f(1)] - \frac{1}{12}\left(\frac{1}{3}f^{(4)}(\xi_2) - \frac{1}{5}f^{(4)}(\xi_1)\right)$$

$$\Rightarrow \left|\int_{-1}^1 f(x)dx - \left(\frac{1}{3}f(-1)+\frac{4}{3}f(0)+\frac{1}{3}f(1)\right)\right| = \frac{1}{90}\left|\frac{5}{2}f^{(4)}(\xi_2) - \frac{3}{2}f^{(4)}(\xi_1)\right|$$

$$\leq \frac{1}{90}\left(\frac{5}{2}+\frac{3}{2}\right)\max_{-1\leq\xi\leq1}|f^{(4)}(\xi)|$$

$$= \frac{2}{45}\max_{-1\leq\xi\leq1}|f^{(4)}(\xi)|$$

3. To get largest degree of precision, a, b, c have to satisfy the following

equations: $\begin{cases} 2a+b = \int_{-1}^{1} 1\,dx = 2 \\ 2ac^2 = \int_{-1}^{1} x^2\,dx = \frac{2}{3} \\ 2ac^4 = \int_{-1}^{1} x^4\,dx = \frac{2}{5} \end{cases}$

Since $\int_{-1}^{1} f(x) = a(f(c)+f(-c)) + bf(0)$ would be satisfied directly if $f(x)=x^5$

Moreover, when $f(x)=x^6$, if the equality holds $\Rightarrow 2ac^6 = \frac{2}{7}$

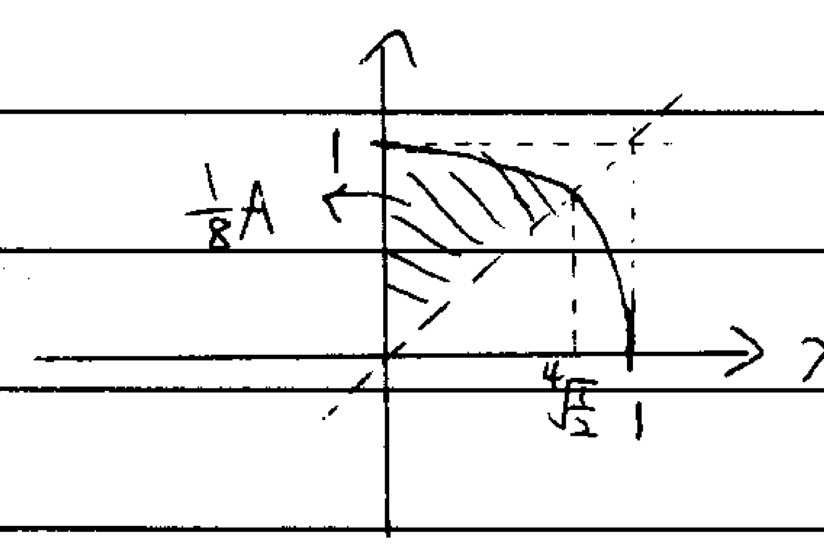
$\Rightarrow \begin{cases} 2ac^2 = \frac{2}{3} \\ 2ac^4 = \frac{2}{5} \\ 2ac^6 = \frac{2}{7} \end{cases}$ $\begin{matrix} \Rightarrow c^2 = \frac{3}{5} \\ \Rightarrow c^2 = \frac{5}{7} \end{matrix}$ would get a contradiction

Hence, the actual smallest integer $p=6$

4. Let A be the area enclosed by $x^4+y^4=1$, it is obvious that

A is symmetric to $x=0$, $y=0$ and $y=x$.

If $y=x \Rightarrow 2x^4=1 \Rightarrow x=\sqrt[4]{\frac{1}{2}} \Rightarrow$



Hence $\frac{1}{8}A = \int_0^{\sqrt[4]{\frac{1}{2}}} (1-x^4)^{\frac{1}{4}}\,dx - (\sqrt[4]{\frac{1}{2}})^2 \times \frac{1}{2}$

Let $f(x) = (1-x^4)^{\frac{1}{4}} \Rightarrow f^{(4)}(x) = (-6-135x^4-90x^8)(1-x^4)^{-\frac{15}{4}}$

$\Rightarrow \max_{0 \le x \le \sqrt[4]{\frac{1}{2}}} |f^{(4)}(x)| \le (6+135\cdot\frac{1}{2}+90\cdot\frac{1}{4})\cdot(\frac{1}{2})^{-\frac{15}{4}} = 96\cdot 2^{\frac{15}{4}}$

Since $0.1 \le \frac{1}{8}A \le 1$, to get 10 correct digits by Simpson's rule

$\Rightarrow \frac{\sqrt[4]{\frac{1}{2}}}{180}\cdot\left(\frac{\sqrt[4]{\frac{1}{2}}}{n}\right)^4 \cdot 96 \cdot 2^{\frac{15}{4}} \le 10^{-11} \times \frac{1}{8}$

$$\Rightarrow n \geq \sqrt[4]{\frac{1}{180} \times 96 \times 8 \times 2^{\frac{5}{2}} \times 10^{11}} = 1246.4\cdots$$

choose $n = 1250 \Rightarrow A = \underline{3.708149354602}049$

check! $n = 2500 \Rightarrow A = \underline{3.708149354602}700$

5. Let $P_{2n}(x) = (x-x_1)^2 \cdot (x-x_2)^2 \cdots (x-x_n)^2 \Rightarrow \deg(P_{2n}) = 2n$

Since $Q(P_{2n}) = \sum_{i=1}^{n} c_i P_{2n}(x_i) = 0$, but $\int_a^b P_{2n}(x)dx$ is obvious larger than 0

Hence the formula can't have degree of precision greater than $2n-1$, regardless of the choices of $c_1, \cdots, c_n$ and $x_1, \cdots, x_n$

6. (a) Partial pivoting:

$$\begin{bmatrix} 1 & -5 & 1 \\ 10 & 0 & 20 \\ 5 & 0 & -1 \end{bmatrix} \xrightarrow[E_1 \leftrightarrow E_2]{} \begin{bmatrix} 10 & 0 & 20 \\ 1 & -5 & 1 \\ 5 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 10 & 0 & 20 \\ 0 & -5 & -1 \\ 0 & 0 & -11 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Scaled partial pivoting:

$$\begin{array}{c} \frac{1}{5} \\ \frac{1}{2} \\ 1 \end{array} \begin{bmatrix} 1 & -5 & 1 \\ 10 & 0 & 20 \\ 5 & 0 & -1 \end{bmatrix} \xrightarrow[E_1 \leftrightarrow E_3]{} \begin{bmatrix} 5 & 0 & -1 \\ 10 & 0 & 20 \\ 1 & -5 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 0 & -1 \\ 0 & 0 & 22 \\ 0 & -5 & \frac{6}{5} \end{bmatrix} \xrightarrow[E_2 \leftrightarrow E_3]{} \begin{bmatrix} 5 & 0 & -1 \\ 0 & -5 & \frac{6}{5} \\ 0 & 0 & 22 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

1. For $i,j = 1, \cdots, n$

$L_{ij} = 0$

$U_{ij} = 0$

end

For $i = 1, \cdots, n$

$L_{ii} = 1$

end

For $k = 1, \cdots, n-1$

For $j = k, \cdots, n$

$U_{kj} = A_{kj}$

end

For $i = k+1, \cdots, n$

$L_{ik} = A_{ik} / A_{kk}$

For $j = k+1, \cdots, n$

$A_{ij} = A_{ij} - L_{ik} / U_{kj}$

end

end

end

$U_{nn} = A_{nn}$

4 (Version II)

Let $x = r\cos\theta$, $y = r\sin\theta$ $\Rightarrow r^4(\cos^4\theta + \sin^4\theta) = 1$

$$\Rightarrow r^2 = \frac{1}{\sqrt{1 - \frac{1}{2}\sin^2 2\theta}}$$

$$\int_0^1 \int_0^{\sqrt[4]{1-x^4}} 1 \, dx\,dy = \int_0^{\frac{\pi}{2}} \int_0^r \tilde{r}\, d\tilde{r}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{r^2}{2} d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - \frac{1}{2}\sin^2 2\theta}} d\theta$$

$$\Rightarrow \text{Area} = 2 \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - \frac{1}{2}\sin^2 2\theta}} d\theta = \int_0^{\pi} \frac{1}{\sqrt{1 - \frac{1}{2}\sin^2 2\theta}} d\theta$$

Let $f(\theta)=\frac{1}{\sqrt{1-\frac{1}{2}\sin^2 2\theta}}$

Since f is periodic with period $\frac{\pi}{2}$, i.e. $f(x)=f(x+\frac{\pi}{2})$

$\Rightarrow f'$ is also periodic with period $\frac{\pi}{2}$ $\Rightarrow f'(\pi)-f'(0)=0$

Let T_n be the approximation of $\int_0^{\pi}\frac{1}{\sqrt{1-\frac{1}{2}\sin^2 2\theta}}\,d\theta$ generated by Trapezoidal rule $\Rightarrow \int_0^{\pi}\frac{1}{\sqrt{1-\frac{1}{2}\sin^2 2\theta}}\,d\theta = T_n - \frac{\pi}{12}h^2 f''(\mu)$, where $\mu \in (0,\pi)$

Moreover, $-\frac{h^2}{12}f''(\mu) \longrightarrow -\frac{h^2}{12}\underbrace{[f'(\pi)-f'(0)]}_{0}$ as $h\to 0$

Hence, the order of convergence is not just 2 but more rapidly.

choose $n=25 \Rightarrow T_{25}=3.708149354602\cdots$

check! $n=26 \Rightarrow T_{26}=3.70814935447\cdots$

$n=27 \Rightarrow T_{27}=3.708149354602\cdots$

Their first 10 digits are all the same, we can almost sure the answer is correct.

More detail would be showed in txt file.