Quiz 06

1. $\int_{-1}^{1} f(x)\,dx = a f(-1) + b f(1) + c f'(-1) + d f'(1)$

$f(x)=1 \Rightarrow a+b+c+d=2$

$f(x)=x \Rightarrow -a+b+c+d=0$

$f(x)=x^2 \Rightarrow a+b-2c+2d=\frac{2}{3}$

$f(x)=x^3 \Rightarrow -a+b+3c+3d=0$

$$\Rightarrow \begin{cases} a=1 \\ b=1 \\ c=\frac{1}{3} \\ d=-\frac{1}{3} \end{cases}$$

2. $\int_{-1}^{1} f(x)\,dx = c_1 f(x_1) + c_2 f(x_2)$

$f(x)=1 \Rightarrow c_1+c_2=2 \cdots ①$

$f(x)=x \Rightarrow c_1x_1+c_2x_2=0 \cdots ②$

$f(x)=x^2 \Rightarrow c_1x_1^2+c_2x_2^2=\frac{2}{3} \cdots ③$

$f(x)=x^3 \Rightarrow c_1x_1^3+c_2x_2^3=0 \cdots ④$

② $\Rightarrow c_1x_1=-c_2x_2$ 代入 ③,④

$$\Rightarrow \begin{cases} c_2x_2(x_2-x_1)=\frac{2}{3} \\ c_2x_2(x_2^2-x_1^2)=0 \end{cases} \Rightarrow x_1=-x_2\neq 0$$

② $\Rightarrow (2-c_2)(-x_2)+c_2x_2=0$

$\Rightarrow 2x_2(c_2-1)=0 \Rightarrow c_2=1 \Rightarrow c_1=1$

③ $\Rightarrow 2x_2^2=\frac{2}{3} \Rightarrow \begin{cases} x_1=-\sqrt{\frac{1}{3}} \\ x_2=\sqrt{\frac{1}{3}} \end{cases} \Rightarrow \int_{-1}^{1} f(x)\,dx = f(-\sqrt{\tfrac{1}{3}}) + f(\sqrt{\tfrac{1}{3}})$

3. Find order of convergence of Midpoint rule applied directly to the integral $\int_0^1 x^{-\frac{1}{3}}\,dx$

Numerically:

$h=0.001 \Rightarrow M_{1000}=1.497470045712116$

$h=0.0005 \Rightarrow M_{2000}=1.498406223391481$

$h=0.00025 \Rightarrow M_{4000} = 1.4989959823317742$

$\Rightarrow \log_2 \frac{M_{1000}-M_{2000}}{M_{2000}-M_{4000}} = 0.6666569845832397 \approx \frac{2}{3}$

Analytically:

Midpoint rule: $h\left(\sum_{i=1}^{n} f\left(\left(i-\frac{1}{2}\right)h\right)\right)$

Since $\int_{\frac{h}{2}}^{1} x^{-\frac{1}{3}}dx \le h\left(\sum_{i=1}^{n} f\left(\left(i-\frac{1}{2}\right)h\right)\right) \le \int_{\frac{h}{2}}^{1} x^{-\frac{1}{3}}dx + \left(\frac{h}{2}\right)^{-\frac{1}{3}} \cdot h$

$\Rightarrow \frac{3}{2} - \left(\frac{h}{2}\right)^{\frac{2}{3}} \le h\left(\sum_{i=1}^{n} f\left(\left(i-\frac{1}{2}\right)h\right)\right) \le \frac{3}{2} - \left(\frac{h}{2}\right)^{\frac{2}{3}} + 2^{\frac{1}{3}} \cdot h^{\frac{2}{3}}$

$\Rightarrow -2^{-\frac{2}{3}} h^{\frac{2}{3}} \le h\left(\sum_{i=1}^{n} f\left(\left(i-\frac{1}{2}\right)h\right)\right) - \int_0^1 x^{-\frac{1}{3}}dx \le 2^{-\frac{2}{3}} h^{\frac{2}{3}}$

Hence, the order of convergence is $\frac{2}{3}$

4. Find 10 correct digits of $\int_0^1 \frac{e^x}{\sqrt{x}}dx$

Let $f(x) = \frac{e^x}{\sqrt{x}} \Rightarrow f''(x) = \left(\frac{3}{4x^2} - \frac{1}{2x} + 1\right)\frac{e^x}{\sqrt{x}}$ is unbounded in $(0,1)$

Hence we can not directly apply Trapezoidal rule on the integral.

Let $x = t^2 \Rightarrow dx = 2t\,dt \Rightarrow \int_0^1 \frac{e^x}{\sqrt{x}}dx = \int_0^1 2e^{t^2}dt$

Let $g(t) = 2e^{t^2} \Rightarrow g''(t) = (4+8t^2)e^{t^2} \Rightarrow \max_{0\le t\le 1}|g''(t)| \le 12e$

Since $2 \le \int_0^1 2e^{t^2}dt \le 2e \doteqdot 5.4366$

To get 10 correct digits $\Rightarrow \left|-\frac{1-0}{12}\left(\frac{1-0}{n}\right)^2 \cdot 12e\right| \le 10^{-10}$

$\Rightarrow n \ge 164872.127\cdots$

Choose $n=164873$ and use Trapezoidal rule

$\Rightarrow \int_0^1 \frac{e^x}{\sqrt{x}}dx \approx 2.925303491847692$

5.

$\sum_{i=1}^{n-1}[(n-i)+(n-i)^2] = \sum_{i=1}^{n-1}[n-i+n^2-2ni+i^2]$

$= n(n-1) - \frac{n\cdot(n-1)}{2} + n^2(n-1) - 2n\frac{n\cdot(n-1)}{2} + \frac{(n-1)n(2n-1)}{6}$

$= \frac{1}{3}n^3 - \frac{1}{3}n$

Leading order : $\frac{1}{3}n^3$

4. Find 10 correct digits of $\int_0^1 \frac{e^x}{\sqrt{x}}dx = \int_0^1 2e^{t^2}dt$ by Simpson's rule

Let $g(t)=2e^{t^2} \Rightarrow g'''(t)=(24t+16t^3)e^{t^2} \Rightarrow g'''(1)-g'''(0)=40e$

Let E be the error for Composite Simpson's rule

According to exercise 4.4-16 $\Rightarrow E \to -\frac{h^4}{180}[g'''(b)-g'''(a)]$ as $h \to 0$

To get 10 correct digits $\Rightarrow \left|-\frac{1}{180}\left(\frac{1}{n}\right)^4 \cdot 40e\right| \le 10^{-10}$

$\Rightarrow n \ge 278.78\cdots$

$n=300 \Rightarrow S_{300} = \underline{2.925303491}888938$

$n=600 \Rightarrow S_{600} = \underline{2.925303491}819025$

$n=1200 \Rightarrow S_{1200} = \underline{2.925303491}814653$

Since their 10 digits are all the same, we can estimate 10 correct digits of $\int_0^1 \frac{e^x}{\sqrt{x}} dx$ is 2.925303491