Quiz 05

1. Composite Trapezoidal rule : $\int_a^b f(x)dx = \frac{h}{2}[f(a)+2\sum_{j=1}^{n-1} f(x_j) + f(b)] - \frac{b-a}{12}h^2 f''(\mu)$

where $\mu \in (a,b)$

$f(x) = e^{x^2} \Rightarrow f'(x) = 2xe^{x^2} \Rightarrow f''(x) = (2+4x^2)e^{x^2}$

To bound the error within $10^{-6} \Rightarrow \frac{1}{12}h^2 f''(\mu) < 10^{-6}$

Since $\max_{\mu\in(0,1)} f''(\mu) = 6e \Rightarrow h^2 < \frac{2\times10^{-6}}{e} \Rightarrow h < \sqrt{\frac{2}{e}}\times10^{-3} \Rightarrow n > 1165.8219\cdots$

Hence we choose $n=1166$ to get our numerical answer

$\Rightarrow \int_0^1 e^{x^2}dx \approx 1.462652079$

2. Simpson's rule : $\int_{x_0}^{x_2} f(x)\,dx \approx \frac{h}{3}[f(x_0)+4f(x_1)+f(x_2)]$

take $x_0=-1$, $x_1=0$, $x_2=1 \Rightarrow h=1$

If $f(x)=1 \Rightarrow \frac{1}{3}[1+4+1]=2=\int_{-1}^1 1\,dx$

If $f(x)=x \Rightarrow \frac{1}{3}[-1+4\cdot0+1]=0=\int_{-1}^1 x\,dx$

If $f(x)=x^2 \Rightarrow \frac{1}{3}[1+4\cdot0+1]=\frac{2}{3}=\int_{-1}^1 x^2dx$

If $f(x)=x^3 \Rightarrow \frac{1}{3}[-1+4\cdot0+1]=0=\int_{-1}^1 x^3dx$

If $f(x)=x^4 \Rightarrow \frac{1}{3}[1+4\cdot0+1]=\frac{2}{3}$, but $\int_{-1}^1 x^4dx = \frac{2}{5} \neq \frac{2}{3}$

Hence the degree of precision for Simpson's Rule is 3

3. Let $\bar{x} = \frac{a+b}{2}$ and $f(x) = f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + \frac{f''(\xi(x))}{2!}(x-\bar{x})^2$

where $\xi(x)$ is between x & $\bar{x}$

$$\int_a^b f(x)\,dx = \int_a^b f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + \frac{f''(\xi(x))}{2}(x-\bar{x})^2\,dx$$

$$= (b-a)f(\bar{x}) + \left[\tfrac{1}{2}(b-\bar{x})^2 - \tfrac{1}{2}(a-\bar{x})^2\right] f'(\bar{x}) + \int_a^b \frac{f''(\xi(x))}{2}(x-\bar{x})^2\,dx$$

Since $\frac{1}{2}(b-\bar{x})^2 - \frac{1}{2}(a-\bar{x})^2 = \frac{1}{2}\left(\frac{b-a}{2}\right)^2 - \frac{1}{2}\left(\frac{a-b}{2}\right)^2 = 0$

and $(x-\bar{x})^2 \geq 0$ for any $x \in \mathbb{R}$, by Weighted Mean Value Theorem for

integrals, we have $\int_a^b \frac{f''(\xi(x))}{2}(x-\bar{x})^2\,dx = \frac{f''(\xi)}{2}\int_a^b (x-\bar{x})^2\,dx$, where $\xi \in (a,b)$

$$\Rightarrow \int_a^b f(x)\,dx - (b-a)f(\bar{x}) = \frac{f''(\xi)}{2}\int_a^b (x-\bar{x})^2\,dx = \frac{f''(\xi)}{2}\left[\frac{(b-\bar{x})^3}{3} - \frac{(a-\bar{x})^3}{3}\right]$$

$$= \frac{f''(\xi)}{24}(b-a)^3 = \frac{h^3}{3}f''(\xi) \qquad (b-a=2h)$$

Hence, $\int_a^b f(x)\,dx = (b-a)f(\bar{x}) + \frac{h^3}{3}f''(\xi)$, where $\xi \in (a,b)$