Quiz 03

1. $(x_0, y_0) = (0,1)$, $(x_1, y_1) = (1,1)$, $(x_2, y_2) = (1.5, 2)$, $(x_3, y_3) = (2,3)$

$\Rightarrow f[x_0, x_1] = \frac{1-1}{1-0} = 0$, $f[x_1, x_2] = \frac{2-1}{1.5-1} = 2$, $f[x_2, x_3] = \frac{3-2}{2-1.5} = 2$

$\Rightarrow f[x_0, x_1, x_2] = \frac{2-0}{1.5-0} = \frac{4}{3}$, $f[x_1, x_2, x_3] = \frac{2-2}{2-1} = 0$

$\Rightarrow f[x_0, x_1, x_2, x_3] = \frac{0-\frac{4}{3}}{2-0} = -\frac{2}{3}$

$\Rightarrow P_3(x) = 1 + 0\cdot(x-0) + \frac{4}{3}(x-0)(x-1) - \frac{2}{3}(x-0)(x-1)(x-1.5)$

$= 1 + \frac{4}{3}x(x-1) - \frac{2}{3}x(x-1)(x-1.5)$

3.

$P_3(x) = 1\cdot\frac{(x-0)(x-2)(x-3)}{(5-0)(5-2)(5-3)} = \frac{1}{30}x(x-2)(x-3)$

4.

Since $\deg(L_{3,i}(x)) = 3$, $i = 0,1,2,3$

$\Rightarrow \deg(f(x)) \leq 3$

Let $g(x) = f(x) - 1 \Rightarrow g(x_0) = g(x_1) = g(x_2) = g(x_3) = 0$ and $\deg(g) \leq 3$

$\Rightarrow g(x) = 0$, $\forall x \in \mathbb{R} \Rightarrow f(x) = 1$, $\forall x \in \mathbb{R}$

Hence, there's only one function $f$ such that

$L_{3,0}(x) + L_{3,1}(x) + L_{3,2}(x) + L_{3,3}(x) = f(x) = 1$, $\forall x \in \mathbb{R}$

5.

$$\max_{x\in[0,1]} |f(x)-p(x)| = \max_{x\in[0,1]} \left| \frac{f^{(4)}(\xi)}{4!}(x-0)(x-\tfrac{1}{3})(x-\tfrac{2}{3})(x-1)\right| \text{, where } \xi\in[0,1]$$

Claim: $\left|x(x-\frac{1}{3})(x-\frac{2}{3})(x-1)\right| \le 3!\left(\frac{1}{3}\right)^4$, $\forall x\in[0,1]$

$\forall x\in[0,1]$, $x_{j-1}\le x\le x_j$ for some $1\le j\le 3$

$$\Rightarrow \left|(x-x_{j-1})(x-x_j)\right|\le\left(\tfrac{1}{3}\right)^2 \text{ and } \begin{cases} |x-x_{j-1-k}|\le(k+1)(\frac{1}{3}), & 1\le k\le j-1 \\ |x-x_{j+t}|\le(t+1)(\frac{1}{3})\le(t+j)(\frac{1}{3}), & 1\le t\le 3-j \end{cases}$$

$$\Rightarrow \left|x(x-\tfrac{1}{3})(x-\tfrac{2}{3})(x-1)\right|\le 3!\left(\tfrac{1}{3}\right)^4,\ \forall x\in[0,1]$$

Hence, $\max_{x\in[0,1]}|f(x)-p(x)|\le \frac{e}{4}\cdot\left(\frac{1}{3}\right)^4$