Midterm 02

1. True

Since $\max_{0\le x\le 1}|\cos x - P_n(x)| = \max_{0\le x\le 1}\left|\frac{\cos^{(n+1)}(\xi_n)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)\right|$ where ξ_n is between x_0 & x_n

and $x_0, x_1, \cdots, x_n$ are uniformly spaced nodes on $[0,1]$

$$\Rightarrow \max_{0\le x\le 1}|\cos x - P_n(x)| \le \frac{n!}{(n+1)!}h^{n+1}, \text{ where } h=\frac{1}{n}<1$$

$$= \frac{1}{n+1}h^{n+1} \to 0 \text{ as } n\to\infty$$

2.

Since $P_3(x) = 1+4x+4x(x-0.25)+\frac{16}{3}x(x-0.25)(x-0.5)$

$\Rightarrow f[x_0,x_1,x_2]=4$ and $f[x_0,x_1,x_2,x_3]=\frac{16}{3}$

$\Rightarrow f[x_1,x_2,x_3] = 4+\frac{16}{3}(0.75-0) = 8$

3.

Let $P_n(x)$ be the degree n interpolating polynomial of f with nodes $x_0,\cdots,x_n$,

Since $i_0, i_1, \cdots, i_n$ is a rearrangement of $0,1,\cdots,n$ and the uniqueness of P_n $\Rightarrow$ $P_n(x) = f(x_0)+f[x_0,x_1](x-x_0)+\cdots+f[x_0,x_1,\cdots,x_n](x-x_0)(x-x_1)\cdots(x-x_{n-1})$

$$= f(x_{i_0})+f[x_{i_0},x_{i_1}](x-x_{i_0})+\cdots+f[x_{i_0},x_{i_1},\cdots,x_{i_n}](x-x_{i_0})(x-x_{i_1})\cdots(x-x_{i_{n-1}})$$

In these two form, coefficients of their nth degree must be the same $\Rightarrow f[x_0,x_1,\cdots,x_n] = f[x_{i_0},x_{i_1},\cdots,x_{i_n}]$

4.

The degree of the Hermite polynomial for the data $(x_0, f(x_0), f'(x_0))$, $(x_1, f(x_1), f'(x_1)), \cdots, (x_n, f(x_n), f'(x_n))$ is $2n+1$

Uniqueness: Let $H_{2n+1}(x)$ be the Hermite polynomial and $P(x)$ be another polynomial satisfying all conditions and $\deg(P)=2n+1$

Let $D(x)=H_{2n+1}(x)-P(x) \Rightarrow \deg(D)\le 2n+1$

Since $D(x_i)=H_{2n+1}(x_i)-P(x_i)=0$, $i=0,1,2,\cdots,n \Rightarrow D'(\xi_i)=0$ for some $x_{i-1}<\xi_i<x_i$, $i=1,\cdots,n$

and $D'(x_i)=H'_{2n+1}(x_i)-P'(x_i)=0$, $i=0,1,2,\cdots,n$

Hence $D'(x)$ has $2n+1$ zeros

But $\deg(D')\le 2n \Rightarrow D'(x)=0,\ \forall x\in\mathbb{R} \Rightarrow D(x)=\text{constant},\ \forall x\in\mathbb{R}$

Since $D(x_0)=0 \Rightarrow D(x)=0,\ \forall x\in\mathbb{R} \Rightarrow H_{2n+1}(x)=P(x),\ \forall x\in\mathbb{R}$

5.

Since $P(-1)=P'(-1)=P(1)=P'(1)=0$

Assume $P(x)=(ax^2+bx+c)(x-1)^2(x+1)^2=(ax^2+bx+c)(x^2-1)^2$

$$\Rightarrow \begin{cases} P'(x)=(2ax+b)(x^2-1)^2+4x(x^2-1)(ax^2+bx+c) \\ P''(x)=2a(x^2-1)^2+4x(x^2-1)(2ax+b)+(12x^2-4)(ax^2+bx+c)+4x(x^2-1)(2ax+b) \end{cases}$$

$P(0)=1 \Rightarrow c=1$, $P'(0)=2 \Rightarrow b=2$, $P''(0)=3 \Rightarrow 2a-4c=3 \Rightarrow a=\frac{7}{2}$

$\Rightarrow P(x)=(\frac{7}{2}x^2+2x+1)(x^2-1)^2$

6.

To construct a piecewise polynomial interpolation $S(x)$ on the data $(x_0,f(x_0)),(x_1,f(x_1)),\cdots,(x_n,f(x_n))$ with additional continuity conditions

For S', S'' and S''' on the interior nodes $x_1, \cdots, x_{n-1}$, we have to satisfy following equations: assume $S(x) = S_i(x)$ when $x_{i-1} \le x \le x_i$, $i = 1, 2, \cdots, n$

(i) $S_i(x_{i-1}) = f(x_{i-1})$, $S_i(x_i) = f(x_i)$, $i = 1, 2, \cdots, n$

$\Rightarrow 2n$ equations

(ii) $S_i'(x_i) = S_{i+1}'(x_i)$, $S_i''(x_i) = S_{i+1}''(x_i)$, $S_i'''(x_i) = S_{i+1}'''(x_i)$, $i = 1, 2, \cdots, n-1$

$\Rightarrow 3n-3$ equations

Totally we have $5n-3$ equations and n intervals, hence the minimal degree needed in each interval is 4.

Since we have $5n$ variables, we need three additional end conditions.

7.

$S(0.1) = 0.991078079769$

8.

$$f(x+2h) = f(x) + 2h f'(x) + 2h^2 f''(x) + \frac{4}{3}h^3 f'''(x) + \frac{2}{3}h^4 f^{(4)}(x) + \frac{4}{15}h^5 f^{(5)}(x) + O(h^6)$$
$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \frac{h^4}{24} f^{(4)}(x) + \frac{h^5}{120} f^{(5)}(x) + O(h^6)$$
$$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + \frac{h^4}{24} f^{(4)}(x) - \frac{h^5}{120} f^{(5)}(x) + O(h^6)$$
$$f(x-2h) = f(x) - 2h f'(x) + 2h^2 f''(x) - \frac{4}{3}h^3 f'''(x) + \frac{2}{3}h^4 f^{(4)}(x) - \frac{4}{15}h^5 f^{(5)}(x) + O(h^6)$$

Consider linear combination of these four expression:

$\cancel{a f(x+2h) + b f(x+h) + c f(x-h) + d f(x-2h)}$

(i) For $f'(x)$ with 4th order

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$$\begin{cases} 2a+b-c-2d=1 \\ 2a+\frac{b}{2}+\frac{c}{2}+2d=0 \\ \frac{4}{3}a+\frac{1}{6}b-\frac{1}{6}c-\frac{4}{3}d=0 \\ \frac{2}{3}a+\frac{1}{24}b+\frac{1}{24}c+\frac{2}{3}d=0 \end{cases} \Rightarrow \begin{cases} 2(a-d)+(b-c)=1 \\ 4(a+d)+(b+c)=0 \\ 8(a-d)+(b-c)=0 \\ 16(a+d)+(b+c)=0 \end{cases} \Rightarrow \begin{cases} a-d=-\frac{1}{6} \\ b-c=\frac{4}{3} \\ a+d=0 \\ b+c=0 \end{cases} \Rightarrow \begin{cases} a=-\frac{1}{12} \\ b=\frac{2}{3} \\ c=-\frac{2}{3} \\ d=\frac{1}{12} \end{cases}$$

$$\Rightarrow -\frac{1}{12}f(x+2h)+\frac{2}{3}f(x+h)-\frac{2}{3}f(x-h)+\frac{1}{12}f(x-2h)=hf'(x)+O(h^5)$$

$$\Rightarrow f'(x)=\frac{1}{12h}\left(-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)\right)+O(h^4)$$

(ii) For $f''(x)$ with 4th order

$$\begin{cases} 2a+b-c-2d=0 \\ 2a+\frac{b}{2}+\frac{c}{2}+2d=1 \\ \frac{4}{3}a+\frac{1}{6}b-\frac{1}{6}c-\frac{4}{3}d=0 \\ \frac{2}{3}a+\frac{1}{24}b+\frac{1}{24}c+\frac{2}{3}d=0 \\ \frac{4}{15}a+\frac{1}{120}b-\frac{1}{120}c-\frac{4}{15}d=0 \end{cases} \Rightarrow \begin{cases} 2(a-d)+(b-c)=0 \\ 4(a+d)+(b+c)=2 \\ 8(a-d)+(b-c)=0 \\ 16(a+d)+(b+c)=0 \\ 32(a-d)+(b-c)=0 \end{cases} \Rightarrow \begin{cases} a-d=0 \\ b-c=0 \\ a+d=-\frac{1}{6} \\ b+c=\frac{8}{3} \end{cases} \Rightarrow \begin{cases} a=-\frac{1}{12} \\ b=\frac{4}{3} \\ c=\frac{4}{3} \\ d=-\frac{1}{12} \end{cases}$$

$$\Rightarrow -\frac{1}{12}f(x+2h)+\frac{4}{3}f(x+h)+\frac{4}{3}f(x-h)-\frac{1}{12}f(x-2h)=\frac{5}{2}f(x)+h^2f''(x)+O(h^6)$$

$$\Rightarrow f''(x)=\frac{1}{12h^2}\left(-f(x+2h)+16f(x+h)-30f(x)+16f(x-h)-f(x-2h)\right)=f''(x)+O(h^4)$$

9.

(i) Suppose $\quad M=N(h)+K_1h^{p_1}+K_2h^{p_2}+\cdots \approx N(h)+K_1h^{p_1}$

$$\Rightarrow N(h)-N\left(\tfrac{h}{2}\right)\approx K_1\left(\tfrac{h}{2}\right)^{p_1}(1-2^{p_1})$$

and $N\left(\frac{h}{2}\right)-N\left(\frac{h}{4}\right)\approx K_1\left(\frac{h}{4}\right)^{p_1}(1-2^{p_1})$

$$\Rightarrow \frac{N(h)-N(\frac{h}{2})}{N(\frac{h}{2})-N(\frac{h}{4})}\approx 2^{p_1}$$

$$\Rightarrow p_1\approx \log_2\left(\frac{N(0.2)-N(0.1)}{N(0.1)-N(0.05)}\right)\doteqdot 2.977 \Rightarrow \text{The order is three}$$

(ii) Since $M = N(h) + K_1 h^3 + \cdots$

We can give a better approximation of M than $N(0.05)$ by the following calculation:

$$N_1(0.2) = \frac{8N(0.1) - N(0.2)}{7} \doteqdot 1.23$$

$$N_1(0.1) = \frac{8N(0.05) - N(0.1)}{7} \doteqdot 1.22996$$

$$\Rightarrow N_2(0.2) = \frac{16N_1(0.1) - N_1(0.2)}{15} \doteqdot 1.2299542857$$

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