Midterm 01

1. $\pm 1.a_1a_2\cdots a_s \times 2^e$ with $s=23$, $a_j \in \{0,1\}$, $-127 \le e \le 128$

Since $128+127+1 = 256 = 2^8 \Rightarrow$ we need 8 bits to store "e"

$\pm$ e $a_1 \cdots a_{23}$

$1+8+23 = 32$ (bits)

The largest number of this form : $0\,\overbrace{1111111}^{7\text{ terms}}0\,\overbrace{11\cdots\cdots11}^{23\text{ terms}}$

$= 2^{127}\cdot(2-2^{-23})$

2.

$P(x) = 1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\frac{x^8}{4!}+\frac{x^{10}}{5!}$

$= 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}$, where $z=x^2$ (1 multiplication)

$= \left[\left(\left[\left(\frac{z}{5}+1\right)\frac{z}{4}+1\right]\frac{z}{3}+1\right)\frac{z}{2}+1\right]z+1$ (8 multiplications)

Hence we have 9 multiplications

3.

$$x = \frac{1900 \pm \sqrt{1900^2-4}}{2} = \frac{2}{1900 \mp \sqrt{1900^2-4}}$$

$\Rightarrow x = 5.263159352676125 \times 10^{-4}$ or $1.899999473684065 \times 10^{3}$

4.

$P_0 = 1$, $P_1 = \frac{1}{3}$, $P_n = \frac{10}{3}P_{n-1} - P_{n-2}$, this algorithm is unstable

Let P_n^e be the exact solution and P_n^h be the numerical solution

$\Rightarrow \begin{cases} P_0^e = 1, \ P_1^e = \frac{1}{3} \\ P_n^e = \frac{10}{3}P_{n-1}^e - P_{n-2}^e \end{cases}$ and $\begin{cases} P_0^h = 1, \ P_1^h = \frac{1}{3}(1+\delta) \\ P_n^h = \frac{10}{3}P_{n-1}^h - P_{n-2}^h \end{cases}$, for some $\delta > 0$

Per-Duet

Let $e_n = p_n^h - p_n^e$, $\Rightarrow \begin{cases} e_0 = 0 \ , \ e_1 = \frac{1}{3}\delta \\ e_n = \frac{10}{3} e_{n-1} - e_{n-2} \end{cases} \Rightarrow e_n = c_1 (\frac{1}{3})^n + c_2 3^n$

$\Rightarrow \begin{cases} c_1 + c_2 = e_0 = 0 \\ \frac{1}{3} c_1 + 3 c_2 = e_1 = \frac{1}{3}\delta > 0 \end{cases} \Rightarrow c_2 = \frac{\delta}{8} > 0$

Hence the error grows exponentially in n

5.

(a) Let $x_0 = 0.5$ and $x_{n+1} = g(x_n) = \frac{1}{2} x_n - \sin x_n + 0.005$

(b) $|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$, where $|g'(x)| \leq k < 1$ on $[a,b]$ and g maps $[a,b]$ to $[a,b]$

Hence, we have to find a, b and k

$g(x) = \frac{1}{2}x - \sin x + 0.005 \Rightarrow g'(x) = \frac{1}{2} - \cos x > 0$, $\forall x \in [-1, 1]$

Since $g(1) = \frac{1}{2} - \sin 1 + 0.005 \fallingdotseq -0.3364$ and $g(-1) \fallingdotseq 0.3464$

$\Rightarrow -1 \leq g(x) \leq 1$, $\forall x \in [-1, 1]$

If $g''(x) = \sin x = 0$, $x \in [-1, 1] \Rightarrow x = 0$

Since $g'(-1) = g'(1) \fallingdotseq -0.0403$ and $g'(0) = -\frac{1}{2}$

$\Rightarrow |g'(x)| \leq \frac{1}{2}$, $\forall x \in [-1, 1]$

Hence $a = -1$, $b = 1$; $k = \frac{1}{2}$

$\Rightarrow |p_n - p| \leq (\frac{1}{2})^n \max\{2, 0\} = (\frac{1}{2})^{n-1} < 10^{-6}$

$\Rightarrow n > 1 + \frac{6}{\log_{10} 2} \fallingdotseq 20.93\cdots$

$\Rightarrow n = 21$ is an upper bound for this iteration

6 (a) Let $f(x) = x - g(x)$

(i) Existence

Since $|g'| < \frac{1}{2}$ and $f'(x) = 1 - g'(x)$

$\Rightarrow \frac{1}{2} < f'(x)$

$\Rightarrow f(x) = f(0) + f'(\xi_x)(x - 0)$, where ξ_x is between 0 & x

$\Rightarrow f(x) > f(0) + \frac{1}{2}x$, when $x > 0$

and $f(x) < f(0) + \frac{1}{2}x$, when $x < 0$

Hence, there exist $a > 0$ and $b < 0$, such that

$$\begin{cases} f(a) > 0 \\ f(b) < 0 \end{cases}$$

Therefore, $\exists\ c \in [b, a]$ s.t. $f(c) = 0 \Rightarrow c = g(c)$

(ii) Uniqueness

Suppose $\exists\ x_1 \neq x_2$ s.t. $x_1 = g(x_1)$, $x_2 = g(x_2)$

$\Rightarrow |x_1 - x_2| = |g(x_1) - g(x_2)| = |g'(c)|\,|x_1 - x_2| < \frac{1}{2}|x_1 - x_2|$, where $c \in [x_1, x_2]$

$(\rightarrow\leftarrow)$

Hence $x = g(x)$ has a unique solution

6 (b) Suppose x^* is the unique solution of $x = g(x)$

then $|x_{n+1} - x^*| = |g(x_n) - g(x^*)| = |g'(c_n)|\,|x_n - x^*|$, where c_n is between x_n & x^*

$$< \tfrac{1}{2}|x_n - x^*| < \cdots < \left(\tfrac{1}{2}\right)^{n+1}|x_0 - x^*|$$

which means $x_n \to x^*$ as $n \to \infty$

(c)

Steffensen's method:

Give $x_0^{(0)}$, $\quad x_1^{(n)} = g(x_0^{(n)})$, $\quad x_2^{(n)} = g(x_1^{(n)})$

$$x_0^{(n+1)} = x_0^{(n)} - \frac{(x_1^{(n)} - x_0^{(n)})^2}{x_2^{(n)} - 2x_1^{(n)} + x_0^{(n)}} \qquad n = 0, 1, 2, \cdots$$

7 (a) $f(x) = x^2 - 2x + 1$, $\quad f'(x) = 2x - 2$

Newton's method:

Give x_0, $\quad x_{n+1} = x_n - \dfrac{x_n^2 - 2x_n + 1}{2x_n - 2}$, $n = 0, 1, 2, \cdots$

(b)

The iteration would converge linearly

Let $g(x) = x - \dfrac{f(x)}{f'(x)}$

Since $f(x) = (x-1)^2 \Rightarrow g'(1) = 1 - \frac{1}{2} = \frac{1}{2}$

$\Rightarrow g(x) = g(1) + g'(\xi)(x-1)$, where ξ is between 1 & x

Hence $|x_{n+1} - 1| = |g(x_n) - 1| = |g'(\xi_n)|\,|x_n - 1|$

ξ_n is between 1 & x_n

$\Rightarrow \lim\limits_{n\to\infty} \dfrac{|x_{n+1} - 1|}{|x_n - 1|} = \dfrac{1}{2}$

$\Rightarrow$ The order of convergence $\alpha = 1$

8 (a) $P_n = \cos\frac{1}{n}$

$$\hat{P}_n = P_n - \frac{(P_{n+1}-P_n)^2}{P_{n+2}-2P_{n+1}+P_n} = \cos\frac{1}{n} - \frac{\left(\cos\frac{1}{n+1}-\cos\frac{1}{n}\right)^2}{\cos\frac{1}{n+2}-2\cos\frac{1}{n+1}+\cos\frac{1}{n}}$$

(b)

$$\frac{\hat{P}_n-P}{P_n-P} = \frac{\hat{P}_n-P_n}{P_n-P}+\frac{P_n-P}{P_n-P} = 1-\frac{(P_{n+1}-P_n)^2}{(P_{n+2}-2P_{n+1}+P_n)(P_n-P)} = 1-\frac{[(P_{n+1}-P)-(P_n-P)]^2}{[(P_{n+2}-P)-2(P_{n+1}-P)+(P_n-P)](P_n-P)} \cdots ①$$

Since $P_n-P = \cos\frac{1}{n}-1 = \left(1-\frac{(\frac{1}{n})^2}{2!}\cos(\xi_n)\right)-1 = -\frac{1}{2n^2}\cos(\xi_n)$, where ξ_n is between 0 & $\frac{1}{n}$

$$① \Rightarrow 1-\frac{\left[-\frac{1}{2(n+1)^2}\cos(\xi_{n+1})+\frac{1}{2n^2}\cos(\xi_n)\right]^2}{\left[-\frac{1}{2(n+2)^2}\cos(\xi_{n+2})+\frac{1}{(n+1)^2}\cos(\xi_{n+1})-\frac{1}{2n^2}\cos(\xi_n)\right]\left(-\frac{1}{2n^2}\cos(\xi_n)\right)}$$

$$=1+\frac{[-n^2(n+2)\cos(\xi_{n+1})+(n+1)^2(n+2)\cos(\xi_n)]^2}{[-n^2(n+1)^4\cos(\xi_{n+2})+2n^2(n+1)^2(n+2)^2\cos(\xi_{n+1})-(n+1)^4(n+2)^2\cos(\xi_n)]\cos(\xi_n)}$$

and $\lim_{n\to\infty}\cos(\xi_n) = \cos 0 = 1$ (Since ξ_n is between 0 and $\frac{1}{n}$)

Hence $\lim_{n\to\infty}\frac{\hat{P}_n-P}{P_n-P} = \lim_{n\to\infty} 1+\frac{4n^4+o(n^3)}{-6n^4+o(n^3)} = 1-\frac{2}{3} = \frac{1}{3}$

9.

$f(x)=x^3-2$, $x_0=1$, $x_1=2$, $x_2=3$

$$P(x) = a(x-3)^2+b(x-3)+c \Rightarrow c=f(3)=25$$

and $\begin{cases} 4a-2b+25=-1 \\ a-b+25=6 \end{cases} \Rightarrow \begin{cases} a=6 \\ b=25 \end{cases}$

$$\Rightarrow x_3-x_2 = \frac{-2c}{b+\text{sign}(b)\sqrt{b^2-4ac}} = -\frac{5}{3} \Rightarrow x_3 = x_2-\frac{5}{3} = \frac{4}{3}$$

detail in 5(a)

$f(x) = x + 2\sin x - 0.01 \Rightarrow f(1) > 0$ and $f(-1) < 0$

let $g(x) = 0.01 - 2\sin x \Rightarrow g'(x) = -2\cos x$

$\Rightarrow -2 \le g'(x) \le -1.09, \ \forall x \in [-1,1]$

Let $\tilde{g}(x) = \alpha x + (1-\alpha) g(x) \Rightarrow \tilde{g}'(x) = \alpha + (1-\alpha) g'(x)$

Asking for $|\tilde{g}'(x)| < 1 \Rightarrow -1 < \alpha + (1-\alpha) g'(x) < 1$

$\Rightarrow -1 < \alpha(1 - g'(x)) + g'(x) < 1$

Since $2.09 \le 1 - g'(x) \le 3 \Rightarrow \frac{-1-g'(x)}{1-g'(x)} < \alpha < 1$

Since $\max_{x \in [-1,1]} \frac{-1-g'(x)}{1-g'(x)} \fallingdotseq 0.4785\ldots \Rightarrow 0.48 < \alpha < 1$

Hence, we pick $\alpha = \frac{1}{2} \Rightarrow \tilde{g}(x) = \frac{1}{2}x + \frac{1}{2}g(x) = \frac{1}{2}x - \sin x + 0.005$