Final

1. (a) Let $f(x) = \sin x - \frac{1}{2}(\cos 1 - 1)x^2 \Rightarrow f'(x) = \cos x - (\cos 1 - 1)x$

$$\Rightarrow f''(x) = -\sin x - (\cos 1 - 1)$$

Let $h = \frac{1-0}{n+2} = \frac{1}{n+2} \Rightarrow \int_0^1 f(x)\,dx = 2h\sum_{i=0}^{\frac{n}{2}} f(x_{2i}) + \frac{1-0}{6}h^2 f''(\mu)$, where $\mu \in (0,1)$

Since $f'''(x) = -\cos x < 0$ on $[0,1] \Rightarrow f''(x)$ is decreasing on $[0,1]$

$$\Rightarrow \max_{0 \le x \le 1} |f''(x)| = \max\{|f''(0)|, |f''(1)|\} \doteqdot 0.45969\cdots$$

To bound the error within $10^{-6} \Rightarrow \frac{1}{6}h^2 \cdot 0.46 \le 10^{-6} \Rightarrow h \le 3.611575\cdots \times 10^{-3}$

$$\Rightarrow h \ge 274.88\cdots$$

(b)

$h = 0.01 \Rightarrow M_{100} = 0.5363139765765569$

$h = 0.005 \Rightarrow M_{200} = 0.5363139764927570$

$h = 0.0025 \Rightarrow M_{400} = 0.5363139764875197$

Order of convergence $= \log_2\left(\frac{\text{Exact} - M_{100}}{\text{Exact} - M_{200}}\right) \doteqdot 3.999976\cdots \doteqdot 4$

or

$= \log_2\left(\frac{M_{100} - M_{200}}{M_{200} - M_{400}}\right) \doteqdot 4.000034\cdots \doteqdot 4$

(c) Why is the numerical order greater than 2?

Use Taylor's expansion : $\int_{x_{2i-1}}^{x_{2i+1}} f(x)\,dx = \int_{x_{2i-1}}^{x_{2i+1}} f(x_{2i}) + f'(x_{2i})(x - x_{2i}) + \frac{f''(x_{2i})}{2}(x - x_{2i})^2 + \frac{f'''(x_{2i})}{6}(x - x_{2i})^3 + \frac{f^{(4)}(\xi)}{24}(x - x_{2i})^4\,dx$

$$= 2h f(x_{2i}) + \frac{f''(x_{2i})}{3}h^3 + \frac{f^{(4)}(\xi_{2i})}{60}h^5, \text{ where } \xi_{2i} \in (x_{2i-1}, x_{2i+1})$$

$$\Rightarrow \int_0^1 f(x)\,dx = \underbrace{2h\sum_{i=0}^{\frac{n}{2}} f(x_{2i})}_{\text{Midpoint rule}} + \frac{h^3}{3}\sum_{i=0}^{\frac{n}{2}} f''(x_{2i}) + \frac{h^5}{60}\sum_{i=0}^{\frac{n}{2}} f^{(4)}(\xi_{2i})$$

Now we consider $\frac{h^3}{3}\sum_{i=0}^{\frac{n}{2}} f''(x_{2i})$ and $\frac{h^5}{60}\sum_{i=0}^{\frac{n}{2}} f^{(4)}(\xi_{2i})$

① $2h\sum_{i=0}^{\frac{n}{2}} f''(x_{2i})$: Midpoint rule for $\int_0^1 f''(x)dx$

$$\Rightarrow \int_0^1 f''(x)dx = 2h\sum_{i=0}^{\frac{n}{2}} f''(x_{2i}) + \sum_{i=0}^{\frac{n}{2}} \frac{h^3}{3} f^{(4)}(\xi_{2i}), \text{ where } \xi_{2i}\in(x_{2i-1}, x_{2i+1})$$

$$\Rightarrow 2h\sum_{i=0}^{\frac{n}{2}} f''(x_{2i}) = \int_0^1 f''(x)dx - \sum_{i=0}^{\frac{n}{2}} \frac{h^3}{3} f^{(4)}(\xi_{2i}) \quad (\text{since } f'(1)-f'(0)=0)$$

$$= -\frac{h^2}{6}\left[2h\sum_{i=0}^{\frac{n}{2}} f^{(4)}(\xi_{2i})\right] = O(h^2) \quad (\text{since } 2h\sum_{i=0}^{\frac{n}{2}} f^{(4)}(\xi_{2i}) \longrightarrow f'''(1)-f'''(0)=1-\cos 1 \text{ as } h\to 0)$$

② $\frac{h^5}{60}\sum_{i=0}^{\frac{n}{2}} f^{(4)}(\xi_{2i}) = \frac{h^4}{120}\left[2h\sum_{i=0}^{\frac{n}{2}} f^{(4)}(\xi_{2i})\right] = O(h^4)$ (same as above)

Hence, $\int_0^1 f(x)dx - 2h\sum_{i=0}^{\frac{n}{2}} f(x_{2i}) = \underbrace{\frac{h^2}{6}\left[2h\sum_{i=0}^{\frac{n}{2}} f''(x_{2i})\right]}_{O(h^2)} + \underbrace{\frac{h^5}{60}\sum_{i=0}^{\frac{n}{2}} f^{(4)}(\xi_{2i})}_{O(h^4)} = O(h^4)$

2. $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(\xi(x))}{24}x^4$, where $\xi(x)$ is between 0 & x

$$\int_{-1}^1 f(x)\,dx = 2f(0) + \frac{f''(0)}{3} + \frac{1}{24}\int_{-1}^1 f^{(4)}(\xi(x))x^4 dx$$

By Weighted Mean Value Theorem, $\int_{-1}^1 f^{(4)}(\xi(x))x^4dx = f^{(4)}(\xi_1)\int_{-1}^1 x^4 dx = \frac{2}{5}f^{(4)}(\xi_1)$, $\xi_1\in(-1,1)$

Moreover, $f''(0) = \frac{1}{1^2}[f(-1)-2f(0)+f(1)] - \frac{1^2}{12}f^{(4)}(\xi_2)$, $\xi_2\in(-1,1)$

$$\Rightarrow \int_{-1}^1 f(x)dx = \frac{1}{3}[f(-1)+4f(0)+f(1)] - \frac{1}{12}\left(\frac{1}{3}f^{(4)}(\xi_2) - \frac{1}{5}f^{(4)}(\xi_1)\right)$$

$$\Rightarrow \left|\int_{-1}^1 f(x)dx - \left(\frac{1}{3}f(-1)+\frac{4}{3}f(0)+\frac{1}{3}f(1)\right)\right| = \frac{1}{90}\left|\frac{5}{2}f^{(4)}(\xi_2) - \frac{3}{2}f^{(4)}(\xi_1)\right| = \frac{1}{90}\left|f^{(4)}(\hat{\xi})\right|, \ \hat{\xi}\in(-1,1)$$

$$\le \frac{1}{90}\max_{\xi\in[-1,1]}\left|f^{(4)}(\xi)\right|$$

3. To get largest degree of precision, a,b,c have to satisfy the following equations:

$$\begin{cases} 2a+b=\int_{-1}^{1} 1\,dx = 2 \\ 2ac^2 = \int_{-1}^{1} x^2\,dx = \frac{2}{3} \\ 2ac^4 = \int_{-1}^{1} x^4\,dx = \frac{2}{5} \end{cases}$$

Since $\int_{-1}^{1} f(x) = a\left(f(c)+f(-c)\right)+bf(0)$ would be satisfied directly if $f(x)=x^5$

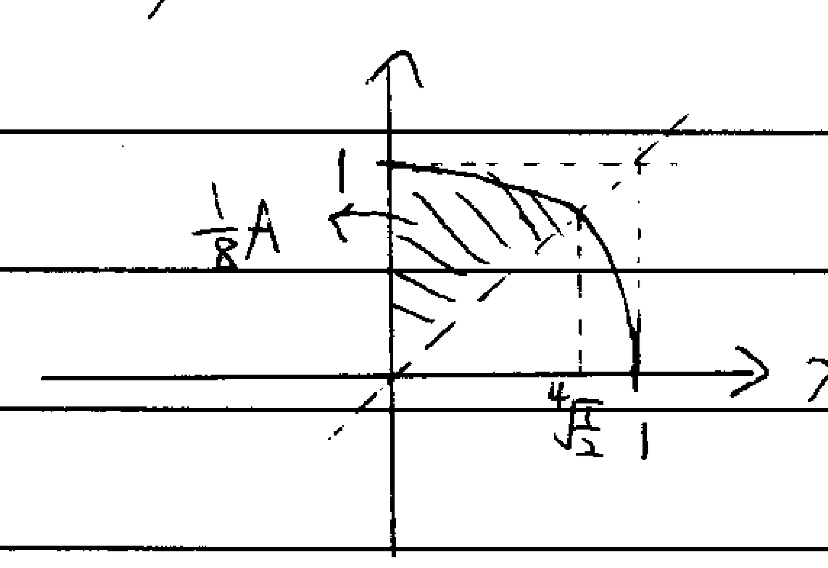
Moreover, when $f(x)=x^6$, if the equality holds $\Rightarrow 2ac^6=\frac{2}{7}$

$$\Rightarrow \begin{cases} 2ac^2=\frac{2}{3} \\ 2ac^4=\frac{2}{5} \\ 2ac^6=\frac{2}{7} \end{cases} \begin{matrix} \Rightarrow c^2=\frac{3}{5} \\ \\ \Rightarrow c^2=\frac{5}{7} \end{matrix}$$ would get a contradiction

Hence, the actual smallest integer $p=6$

4. Let A be the area enclosed by $x^4+y^4=1$, it is obvious that A is symmetric to $x=0$, $y=0$ and $y=x$.

If $y=x \Rightarrow 2x^4=1 \Rightarrow x=\sqrt[4]{\frac{1}{2}} \Rightarrow$



Hence $\frac{1}{8}A=\int_0^{\sqrt[4]{\frac{1}{2}}}(1-x^4)^{\frac{1}{4}}dx-\left(\sqrt[4]{\frac{1}{2}}\right)^2\times\frac{1}{2}$

Let $f(x)=(1-x^4)^{\frac{1}{4}} \Rightarrow f^{(4)}(x)=(-6-135x^4-90x^8)(1-x^4)^{-\frac{15}{4}}$

$$\Rightarrow \max_{0\le x\le \sqrt[4]{\frac{1}{2}}}|f^{(4)}(x)| \le \left(6+135\cdot\frac{1}{2}+90\cdot\frac{1}{4}\right)\cdot\left(\frac{1}{2}\right)^{-\frac{15}{4}} = 96\cdot 2^{\frac{15}{4}}$$

Since $0.1\le\frac{1}{8}A\le 1$, to get 10 correct digits by Simpson's rule

$$\Rightarrow \frac{\sqrt[4]{\frac{1}{2}}}{180}\cdot\left(\frac{\sqrt[4]{\frac{1}{2}}}{n}\right)^4\cdot 96\cdot 2^{\frac{15}{4}} \le 10^{-11}\times\frac{1}{8}$$

$$\Rightarrow n \geq \sqrt[4]{\frac{1}{180} \times 96 \times 8 \times 2^{\frac{5}{2}} \times 10^{11}} = 1246.4\cdots$$

choose $n = 1250 \Rightarrow A = 3.708149354602049$

check! $n = 2500 \Rightarrow A = 3.708149354602700$

5. Let $P_{2n}(x) = (x-x_1)^2 \cdot (x-x_2)^2 \cdots (x-x_n)^2 \Rightarrow \deg(P_{2n}) = 2n$

Since $Q(P_{2n}) = \sum_{i=1}^{n} c_i P_{2n}(x_i) = 0$, but $\int_a^b P_{2n}(x)dx$ is obvious larger than 0

Hence the formula can't have degree of precision greater than $2n-1$, regardless of the choices of $c_1, \cdots, c_n$ and $x_1, \cdots, x_n$

6. (a) Partial pivoting:

$$\begin{bmatrix} 1 & -5 & 1 \\ 10 & 0 & 20 \\ 5 & 0 & -1 \end{bmatrix} \xrightarrow[E_1 \leftrightarrow E_2]{} \begin{bmatrix} 10 & 0 & 20 \\ 1 & -5 & 1 \\ 5 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 10 & 0 & 20 \\ 0 & -5 & -1 \\ 0 & 0 & -11 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Scaled partial pivoting:

$$\begin{array}{c} \frac{1}{5} \\ \frac{1}{2} \\ 1 \end{array} \begin{bmatrix} 1 & -5 & 1 \\ 10 & 0 & 20 \\ 5 & 0 & -1 \end{bmatrix} \xrightarrow[E_1 \leftrightarrow E_3]{} \begin{bmatrix} 5 & 0 & -1 \\ 10 & 0 & 20 \\ 1 & -5 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 0 & -1 \\ 0 & 0 & 22 \\ 0 & -5 & \frac{6}{5} \end{bmatrix} \xrightarrow[E_2 \leftrightarrow E_3]{} \begin{bmatrix} 5 & 0 & -1 \\ 0 & -5 & \frac{6}{5} \\ 0 & 0 & 22 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

1.
```
For i,j=1,...,n
        L_ij = 0
        U_ij = 0
end
For i=1,...,n
        L_ii = 1
end

For k=1,...,n-1
    For j=k,...,n
        U_kj = A_kj
    end

    For i=k+1,...,n
        L_ik = A_ik/A_kk
        For j=k+1,...,n
            A_ij = A_ij - L_ik/U_kj
        end
    end
end
U_nn = A_nn
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4. (Version II)

Let $x = r\cos\theta,\ y = r\sin\theta \Rightarrow r^4(\cos^4\theta + \sin^4\theta) = 1$

$$\Rightarrow r^2 = \frac{1}{\sqrt{1-\frac{1}{2}\sin^2 2\theta}}$$

$$\int_0^1 \int_0^{\sqrt[4]{1-x^4}} 1\, dx\,dy = \int_0^{\frac{\pi}{2}} \int_0^r \tilde{r}\, d\tilde{r}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{r^2}{2}\, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\frac{1}{2}\sin^2 2\theta}}\, d\theta$$

$$\Rightarrow \text{Area} = 2 \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\frac{1}{2}\sin^2 2\theta}}\, d\theta$$

Let $f(\theta) = \frac{1}{\sqrt{1-\frac{1}{2}\sin^2 2\theta}} = \left(\frac{3}{4}+\frac{1}{4}\cos 4\theta\right)^{-\frac{1}{2}}$

$$\Rightarrow f^{(4)}(\theta) = \left(-\frac{327}{16}+\frac{9}{4}\cos 4\theta+\frac{231}{8}\cos^2 4\theta-\frac{27}{4}\cos^3 4\theta+\frac{1}{16}\cos^4 4\theta\right)\left(\frac{3}{4}+\frac{1}{4}\cos 4\theta\right)^{-\frac{9}{2}}$$

$$\Rightarrow \max_{0\le\theta\le\frac{\pi}{2}} |f^{(4)}(\theta)| \le \frac{467}{8}\cdot 2^{\frac{9}{2}}$$

To get 10 correct digits by Simpson's rule

$$\Rightarrow \frac{\frac{\pi}{2}}{180}\left(\frac{\frac{\pi}{2}}{n}\right)^4 \cdot \frac{467}{8}\cdot 2^{\frac{9}{2}} \le \frac{1}{2}\times 10^{-11}$$

$$\Rightarrow n \ge 1935.6\cdots$$

choose $n = 2000 \Rightarrow A = \underline{3.7081493546}02746$

check! $n = 4000 \Rightarrow A = \underline{3.7081493546}02740$