Fixed-Point Iteration

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Definition

x is called a fixed point of a given function f if $f(x) = x$.

Root-finding problems and fixed-point problems

- Find x^* such that $f(x^*) = 0$.
  Let $g(x) = x - f(x)$. Then $g(x^*) = x^* - f(x^*) = x^*$.
  $\Rightarrow x^*$ is a fixed point for $g(x)$.
- Find x^* such that $g(x^*) = x^*$.
  Define $f(x) = x - g(x)$ so that $f(x^*) = x^* - g(x^*) = x^* - x^* = 0$
  $\Rightarrow x^*$ is a zero of $f(x)$.

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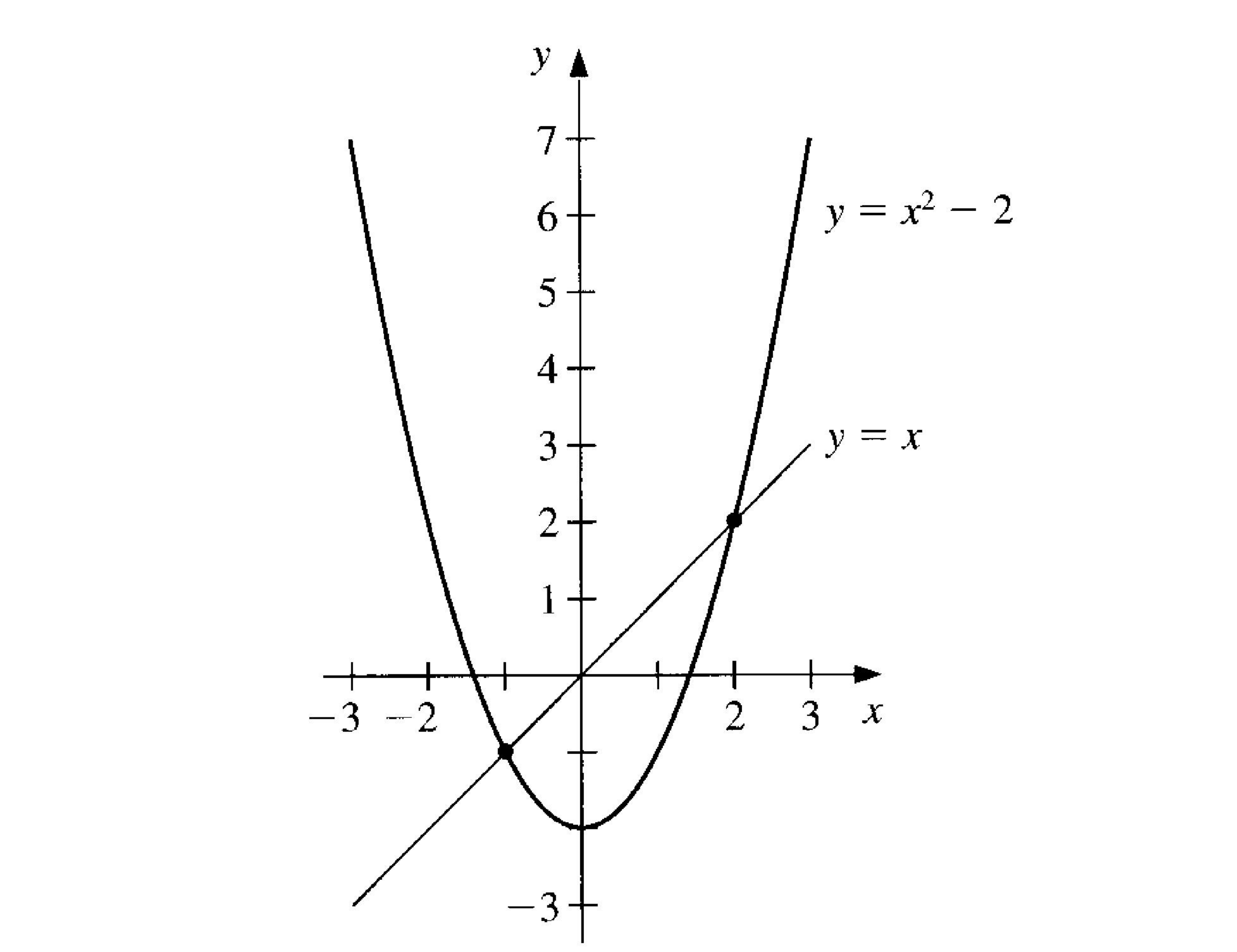
Example

The function $g(x) = x^2 - 2$, for $-2 \leq x \leq 3$, has fixed points at $x = -1$ and $x = 2$ since

$$g(-1) = (-1)^2 - 2 = -1 \quad \text{and} \quad g(2) = 2^2 - 2 = 2.$$



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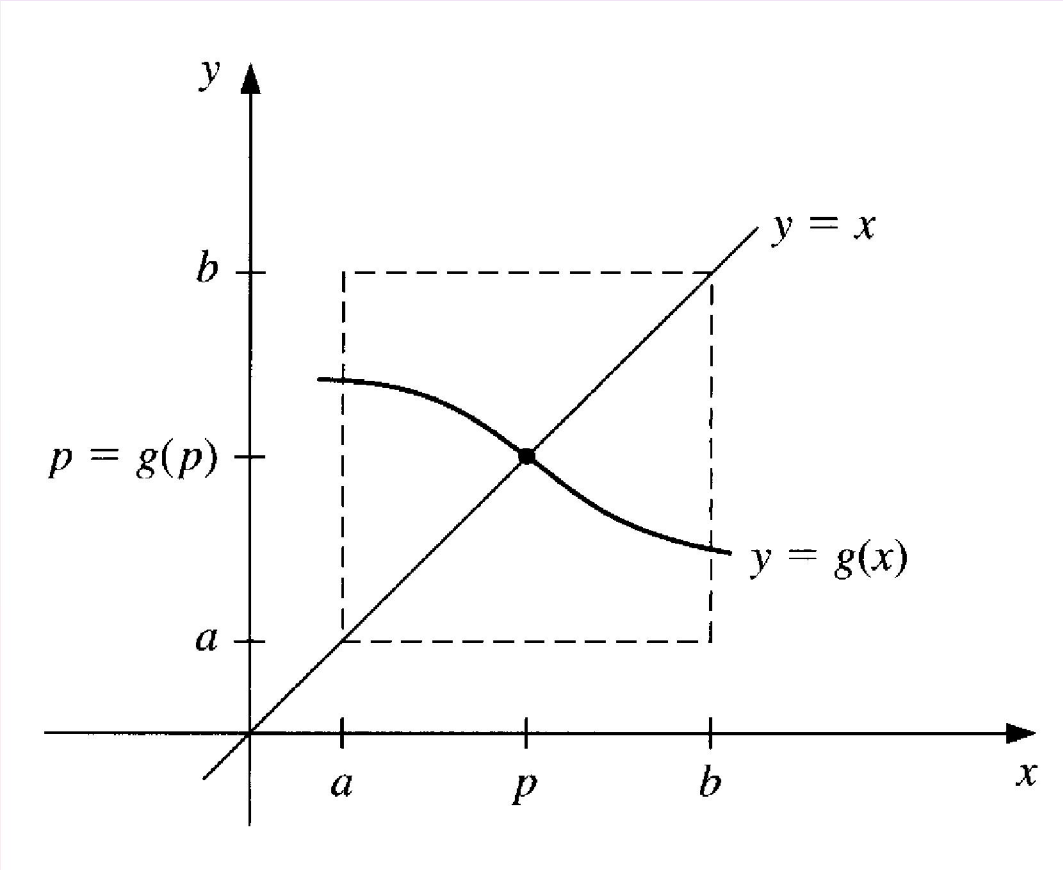
Theorem (Existence and uniqueness)

1. *If $g \in C[a, b]$ such that $a \leq g(x) \leq b$ for all $x \in [a, b]$, then g has a fixed point in $[a, b]$.*
2. *If, in addition, $g'(x)$ exists in (a, b) and there exists a positive constant $M < 1$ such that $|g'(x)| \leq M < 1$ for all $x \in (a, b)$. Then the fixed point is unique.*



Proof

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Existence:

- If $g(a) = a$ or $g(b) = b$, then a or b is a fixed point of g and we are done.
- Otherwise, it must be $g(a) > a$ and $g(b) < b$. The function $h(x) = g(x) - x$ is continuous on $[a, b]$, with

$$h(a) = g(a) - a > 0 \quad \text{and} \quad h(b) = g(b) - b < 0.$$

By the Intermediate Value Theorem, $\exists\ x^* \in [a, b]$ such that $h(x^*) = 0$. That is

$$g(x^*) - x^* = 0 \ \Rightarrow \ g(x^*) = x^*.$$

Hence g has a fixed point x^* in $[a, b]$.

Proof

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Uniqueness:

Suppose that $p \neq q$ are both fixed points of g in $[a, b]$. By the Mean-Value theorem, there exists ξ between p and q such that

$$g'(\xi) = \frac{g(p) - g(q)}{p - q} = \frac{p - q}{p - q} = 1.$$

However, this contradicts to the assumption that $|g'(x)| \leq M < 1$ for all x in $[a, b]$. Therefore the fixed point of g is unique. □

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Example

Show that the following function has a unique fixed point.

$$g(x) = (x^2 - 1)/3, \quad x \in [-1, 1].$$

Solution: The Extreme Value Theorem implies that

$$\min_{x\in[-1,1]} g(x) = g(0) = -\frac{1}{3},$$
$$\max_{x\in[-1,1]} g(x) = g(\pm 1) = 0.$$

That is $g(x) \in [-1, 1], \ \forall \ x \in [-1, 1]$.
Moreover, g is continuous and

$$|g'(x)| = \left|\frac{2x}{3}\right| \le \frac{2}{3}, \ \forall \ x \in (-1, 1).$$

By above theorem, g has a unique fixed point in $[-1, 1]$.

Let p be such unique fixed point of g. Then

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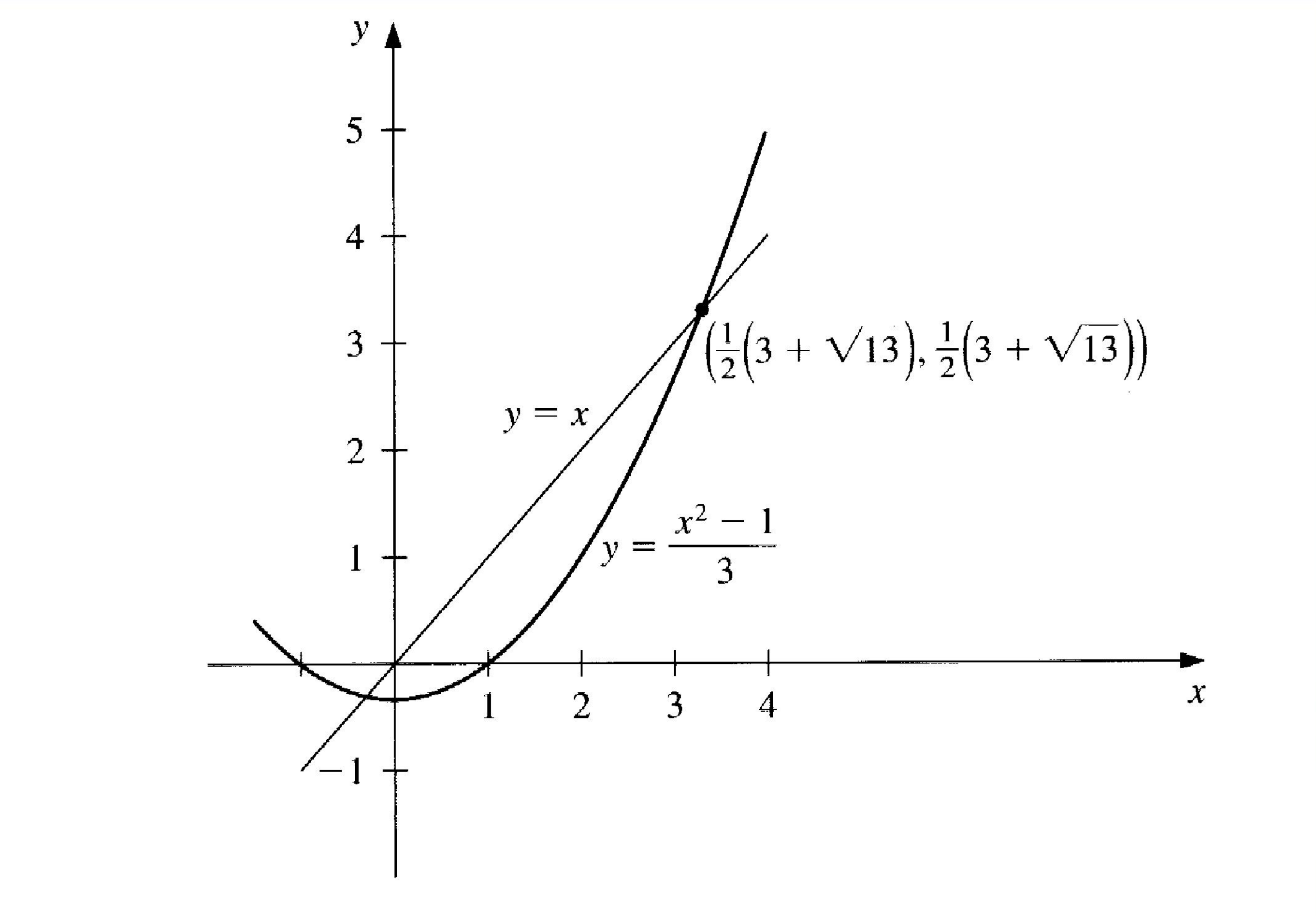
$$p = g(p) = \frac{p^2 - 1}{3} \quad \Rightarrow \quad p^2 - 3p - 1 = 0$$
$$\Rightarrow \quad p = \frac{1}{2}(3 - \sqrt{13}).$$

□



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Fixed-point iteration or functional iteration

Given a continuous function g, choose an initial point x_0 and generate $\{x_k\}_{k=0}^{\infty}$ by

$$x_{k+1} = g(x_k), \quad k \geq 0.$$

$\{x_k\}$ may not converge, e.g., $g(x) = 3x$. However, when the sequence converges, say,

$$\lim_{k\to\infty} x_k = x^*,$$

then, since g is continuous,

$$g(x^*) = g(\lim_{k\to\infty} x_k) = \lim_{k\to\infty} g(x_k) = \lim_{k\to\infty} x_{k+1} = x^*.$$

That is, x^* is a fixed point of g.

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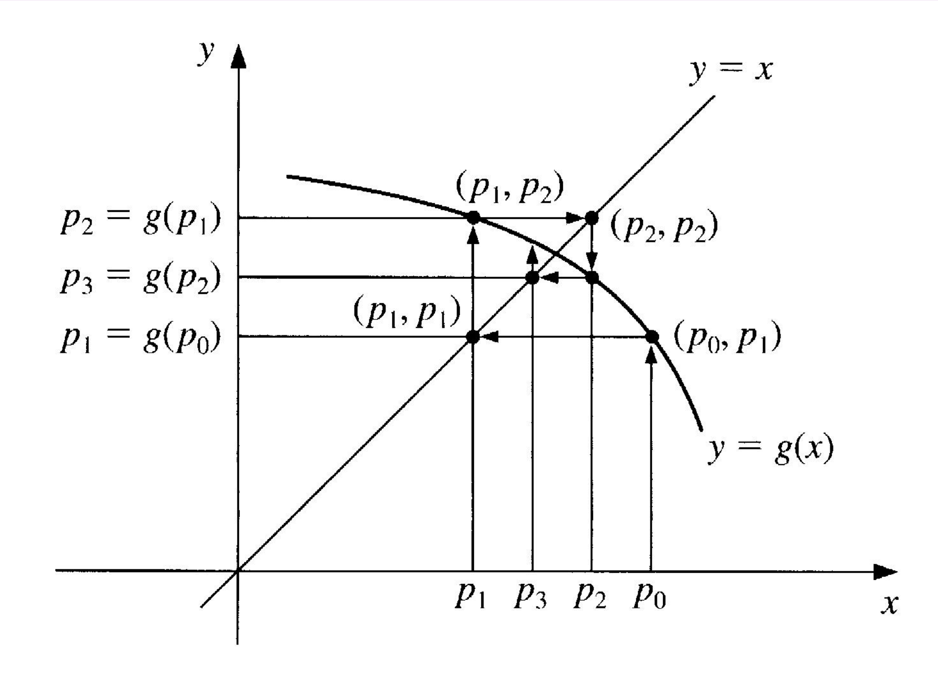
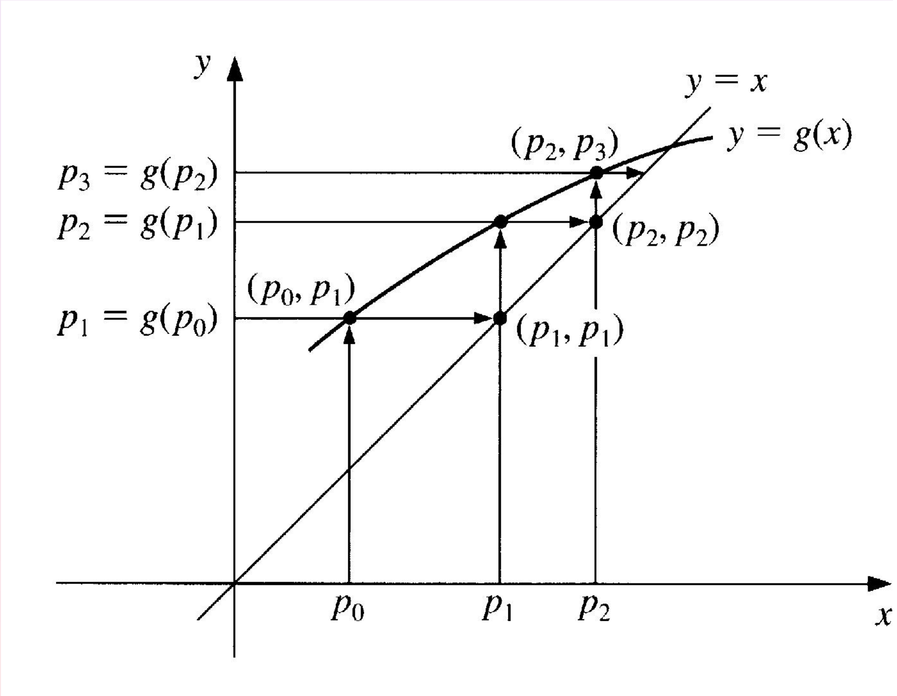
Fixed-point iteration

Given x_0, tolerance TOL, maximum number of iteration M.
Set $i = 1$ and $x = g(x_0)$.
While $i \le M$ and $|x - x_0| \ge TOL$
  Set $i = i + 1$, $x_0 = x$ and $x = g(x_0)$.
End While



(a)



(b)

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Example

The equation

$$x^3 + 4x^2 - 10 = 0$$

has a unique root in $[1, 2]$. Change the equation to the fixed-point form $x = g(x)$.

(a) $x = g_1(x) \equiv x - f(x) = x - x^3 - 4x^2 + 10$

(b) $x = g_2(x) = \left(\frac{10}{x} - 4x\right)^{1/2}$

$$x^3 = 10 - 4x^2 \ \Rightarrow \ x^2 = \frac{10}{x} - 4x \ \Rightarrow \ x = \pm\left(\frac{10}{x} - 4x\right)^{1/2}$$

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(c) $x = g_3(x) = \frac{1}{2}\left(10 - x^3\right)^{1/2}$

$$4x^2 = 10 - x^3 \quad \Rightarrow \quad x = \pm\frac{1}{2}\left(10 - x^3\right)^{1/2}$$

(d) $x = g_4(x) = \left(\frac{10}{4+x}\right)^{1/2}$

$$x^2(x+4) = 10 \quad \Rightarrow \quad x = \pm\left(\frac{10}{4+x}\right)^{1/2}$$

(e) $x = g_5(x) = x - \frac{x^3+4x^2-10}{3x^2+8x}$

$$x = g_5(x) \equiv x - \frac{f(x)}{f'(x)}$$

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Results of the fixed-point iteration with initial point $x_0 = 1.5$

n	(a)	(b)	(c)	(d)	(e)
0	1.5	1.5	1.5	1.5	1.5
1	−0.875	0.8165	1.286953768	1.348399725	1.373333333
2	6.732	2.9969	1.402540804	1.367376372	1.365262015
3	−469.7	$(-8.65)^{1/2}$	1.345458374	1.364957015	1.365230014
4	1.03×10^8		1.375170253	1.365264748	1.365230013
5			1.360094193	1.365225594	
6			1.367846968	1.365230576	
7			1.363887004	1.365229942	
8			1.365916734	1.365230022	
9			1.364878217	1.365230012	
10			1.365410062	1.365230014	
15			1.365223680	1.365230013	
20			1.365230236		
25			1.365230006		
30			1.365230013		

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Theorem (Fixed-point Theorem)

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$ for all $x \in [a, b]$. Suppose that g' exists on (a, b) and that $\exists\ k$ with $0 < k < 1$ such that

$$|g'(x)| \le k, \ \forall\ x \in (a, b).$$

Then, for any number x_0 in $[a, b]$,

$$x_n = g(x_{n-1}), \ n \ge 1,$$

converges to the unique fixed point x in $[a, b]$.

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Proof:

By the assumptions, a unique fixed point exists in $[a, b]$. Since $g([a, b]) \subseteq [a, b]$, $\{x_n\}_{n=0}^{\infty}$ is defined and $x_n \in [a, b]$ for all $n \geq 0$. Using the Mean Values Theorem and the fact that $|g'(x)| \leq k$, we have

$$|x - x_n| = |g(x_{n-1}) - g(x)| = |g'(\xi_n)||x - x_{n-1}| \leq k|x - x_{n-1}|,$$

where $\xi_n \in (a, b)$. It follows that

$$|x_n - x| \leq k|x_{n-1} - x| \leq k^2|x_{n-2} - x| \leq \cdots \leq k^n|x_0 - x|. \tag{1}$$

Since $0 < k < 1$, we have

$$\lim_{n\to\infty} k^n = 0$$

and

$$\lim_{n\to\infty} |x_n - x| \leq \lim_{n\to\infty} k^n|x_0 - x| = 0.$$

Hence, $\{x_n\}_{n=0}^{\infty}$ converges to x.

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Corollary

If g satisfies the hypotheses of above theorem, then

$$|x - x_n| \le k^n \max\{x_0 - a, b - x_0\}$$

and

$$|x_n - x| \le \frac{k^n}{1-k}|x_1 - x_0|, \ \forall \ n \ge 1.$$

Proof: From (1),

$$|x_n - x| \le k^n |x_0 - x| \le k^n \max\{x_0 - a, b - x_0\}.$$

For $n \ge 1$, using the Mean Values Theorem,

$$|x_{n+1} - x_n| = |g(x_n) - g(x_{n-1})| \le k|x_n - x_{n-1}| \le \cdots \le k^n |x_1 - x_0|.$$

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Thus, for $m > n \geq 1$,

$$\begin{aligned}
|x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - \cdots + x_{n+1} - x_n| \\
&\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\
&\leq k^{m-1}|x_1 - x_0| + k^{m-2}|x_1 - x_0| + \cdots + k^n|x_1 - x_0| \\
&= k^n|x_1 - x_0| \left(1 + k + k^2 + \cdots + k^{m-n-1}\right).
\end{aligned}$$

It implies that

$$\begin{aligned}
|x - x_n| &= \lim_{m\to\infty} |x_m - x_n| \leq \lim_{m\to\infty} k^n|x_1 - x_0| \sum_{j=0}^{m-n-1} k^j \\
&\leq k^n|x_1 - x_0| \sum_{j=0}^{\infty} k^j = \frac{k^n}{1-k}|x_1 - x_0|.
\end{aligned}$$

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Example

For previous example, $f(x) = x^3 + 4x^2 - 10 = 0$.

Let $g_1(x) = x - x^3 - 4x^2 + 10$, we have

$$g_1(1) = 6 \quad \text{and} \quad g_1(2) = -12,$$

so $g_1([1,2]) \not\subseteq [1,2]$. Moreover,

$$g_1'(x) = 1 - 3x^2 - 8x \quad \Rightarrow \quad |g_1'(x)| \geq 1 \; \forall \; x \in [1,2]$$

Convergence is NOT guaranteed. In fact, it almost for sure will not converge since when x_n is close to the solution x^*,

$$|x_n - x^*| = |g_1(x_{n-1}) - g_1(x^*)| = |g_1'(c)(x_{n-1} - x^*)| > |x_{n-1} - x^*|.$$

The error is amplified whenever x_n is close to convergence. The only possibility for convergence is when x_n is far from x^* and (by chance, and very unlikely) that $g_1(x_n) = x^*$.

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For $g_3(x) = \frac{1}{2}(10 - x^3)^{1/2}$, $\forall\ x \in [1, 1.5]$,

$$g_3'(x) = -\frac{3}{4}x^2(10 - x^3)^{-1/2} < 0,\ \forall\ x \in [1, 1.5],$$

so g_3 is strictly decreasing on $[1, 1.5]$ and

$$1 < 1.28 \approx g_3(1.5) \le g_3(x) \le g_3(1) = 1.5,\ \forall\ x \in [1, 1.5].$$

On the other hand,

$$|g_3'(x)| \le |g_3'(1.5)| \approx 0.66,\ \forall\ x \in [1, 1.5]$$

Hence, the sequence is convergent to the fixed point.

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For $g_4(x) = \sqrt{10/(4+x)}$, we have

$$\sqrt{\frac{10}{6}} \leq g_4(x) \leq \sqrt{\frac{10}{5}}, \ \forall\ x \in [1,2] \quad \Rightarrow \quad g_4([1,2]) \subseteq [1,2]$$

Moreover,

$$|g_4'(x)| = \left| \frac{-5}{\sqrt{10}(4+x)^{3/2}} \right| \leq \frac{5}{\sqrt{10}(5)^{3/2}} < 0.15, \ \forall\ x \in [1,2].$$

The bound of $|g_4'(x)|$ is much smaller than the bound of $|g_3'(x)|$, which explains the more rapid convergence using g_4.