Numerical Analysis I
Solutions of Equations in One Variable
Section 2.1: Bisection

Instructor: Wei-Cheng Wang [1]

Department of Mathematics
National TsingHua University

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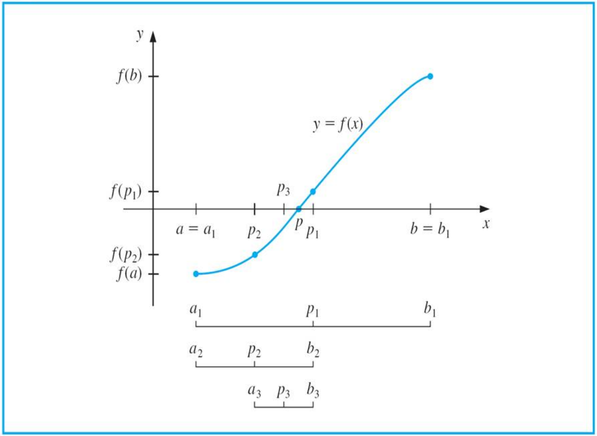
[1] These slides are based on Prof. Tsung-Ming Huang(NTNU)'s original slides

Bisection Method

Idea

If $f(x) \in C[a, b]$ and $f(a)f(b) < 0$, then $\exists\ c \in (a, b)$ such that $f(c) = 0$.

Figure 2.1



Bisection method algorithm

Given $f(x)$ defined on (a, b), the maximal number of iterations M, and stop criteria δ and ε, this algorithm tries to locate one root of $f(x)$.

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Compute fa = f(a), fb = f(b), and e = b − a
If  sign(fa) = sign(fb), then  stop  End If
For k = 1, 2, ..., M
   e = e/2, c = (a + b)/2, fc = f(c)
   If (|e| < δ or |fc| < ε), then  stop  End If
   If  sign(fc) ≠ sign(fa)
      b = c, fb = fc
   Else
      a = c, fa = fc
   End If
End For
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Let $\{c_n\}$ be the sequence of numbers produced. The algorithm should stop if one of the following conditions is satisfied.

1. the iteration number $k > M$,
2. $|c_k - c_{k-1}| < \delta$, or
3. $|f(c_k)| < \varepsilon$.

Let $[a_0, b_0], [a_1, b_1], \ldots$ denote the successive intervals produced by the bisection algorithm. Then

$$\begin{aligned} & a = a_0 \le a_1 \le a_2 \le \cdots \le b_0 = b \\ \Rightarrow \quad & \{a_n\} \text{ and } \{b_n\} \text{ are bounded} \\ \Rightarrow \quad & \lim_{n\to\infty} a_n \text{ and } \lim_{n\to\infty} b_n \text{ exist} \end{aligned}$$

Since

$$\begin{aligned} b_1 - a_1 &= \frac{1}{2}(b_0 - a_0) \\ b_2 - a_2 &= \frac{1}{2}(b_1 - a_1) = \frac{1}{4}(b_0 - a_0) \\ &\vdots \\ b_n - a_n &= \frac{1}{2^n}(b_0 - a_0) \end{aligned}$$

hence

$$\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = \lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} \frac{1}{2^n}(b_0 - a_0) = 0.$$

Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \equiv z.$$

Since f is a continuous function, we have that

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(z) \quad \text{and} \quad \lim_{n\to\infty} f(b_n) = f(\lim_{n\to\infty} b_n) = f(z).$$

On the other hand,

$$\begin{aligned} & f(a_n)f(b_n) < 0 \\ \Rightarrow\quad & \lim_{n\to\infty} f(a_n)f(b_n) = f^2(z) \le 0 \\ \Rightarrow\quad & f(z) = 0 \end{aligned}$$

Therefore, the limit of the sequences $\{a_n\}$ and $\{b_n\}$ is a zero of f in $[a, b]$. Let $c_n = \frac{1}{2}(a_n + b_n)$. Then

$$\begin{aligned} |z - c_n| &= \left| \lim_{n\to\infty} a_n - \frac{1}{2}(a_n + b_n) \right| \\ &= \left| \frac{1}{2}\left[\lim_{n\to\infty} a_n - b_n\right] + \frac{1}{2}\left[\lim_{n\to\infty} a_n - a_n\right] \right| \\ &\le \max\left\{ \left|\lim_{n\to\infty} a_n - b_n\right|, \left|\lim_{n\to\infty} a_n - a_n\right| \right\} \\ &\le |b_n - a_n| = \frac{1}{2^n}|b_0 - a_0|. \end{aligned}$$

This proves the following theorem.

Theorem

Let $\{[a_n, b_n]\}$ *denote the intervals produced by the bisection algorithm. Then* $\lim_{n\to\infty} a_n$ *and* $\lim_{n\to\infty} b_n$ *exist, are equal, and represent a zero of* $f(x)$*. If*

$$z = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \quad \text{and} \quad c_n = \frac{1}{2}(a_n + b_n),$$

then

$$|z - c_n| \leq \frac{1}{2^n}(b_0 - a_0).$$

Remark

$\{c_n\}$ converges to z with the rate of $O(2^{-n})$.

Example

How many steps should be taken to compute a root of $f(x) = x^3 + 4x^2 - 10 = 0$ on $[1, 2]$ with relative error 10^{-3}?

solution: Seek an n such that

$$\frac{|z - c_n|}{|z|} \leq 10^{-3} \;\Rightarrow\; |z - c_n| \leq |z| \times 10^{-3}.$$

Since $z \in [1, 2]$, it is sufficient to show

$$|z - c_n| \leq 10^{-3}.$$

That is, we solve

$$2^{-n}(2 - 1) \leq 10^{-3} \;\Rightarrow\; -n \log_{10} 2 \leq -3$$

which gives $n \geq 10$.